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MONOTONE ITERATIVE METHOD FOR BOUNDARY VALUE PROBLEM WITH LINEAR CONDITION OF FIRST ORDER DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper we discuss the boundary value problem for a first order delay differential equation of the type, $y'(t) + \lambda y(t) = f(t, y(t-r))$. We prove the existence of solution between weakly coupled lower and upper solution by assuming f to be a non-decreasing function in the second coordinate. Further, we use this existence result to establish monotone iterative method, where we obtain increasing as well as decreasing sequence of functions whose limits are a solution of the boundary value problem. The sequence of functions obtained are solutions of some defined boundary value problem with linear condition of linear delay differential equation.

1. Introduction. Boundary value problems (BVPs) for differential equations with deviating arguments (DEDA) have their occurrence in various fields such as production problems in Economics, Biological systems, Physical models etc. So many researchers have strived to get existence of solution for such

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problems. For existence theory on ordinary differential equations (ODEs) as well as delay differential equations (DDE), one may refer to [4, 5, 6]. Many existence results in differential equations are proved using variety of fixed point theorems given in [13]. Existence results on solution between lower and upper solutions for BVP of first order ordinary differential equation are discussed in [8] and [15]. One may refer [3, 9, 18] for BVP of first order DDEs.

In [1], a theorem on existence of solution for Periodic Boundary Value Problem (PBVP) of First Order Delay Differential Equation (FODDE) given by,

$$\begin{aligned} &y'(t) + \lambda y(t) = f(t, y(t-r)),\ t \in I, \\ &y(t) = y(0),\ t \in [-r, 0], \\ &y(0) = y(T),\ \text{is proved, by using Leray–Schauder Alternative.} \end{aligned}$$

Definitions of lower and upper solutions for the PBVP are given to prove a theorem on existence of solution between lower and upper solutions using Schauder's fixed point theorem for the PBVP. Also strict lower and strict upper solutions for the PBVP are defined to prove a theorem on existence of solution strictly between strict lower and strict upper solutions using Schauder's fixed point theorem for the PBVP.

Periodic boundary value problem for first order DDE of the type $x'(t) = f(t, x_t)$, $t \in I = [0, T]$, $x(0) = \phi(0) = x(\phi)(T)$, $x(\theta) = \phi(\theta)$; $\theta \in [-r, 0]$, is studied in [2]. Here $\phi \in C[-r, 0]$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $0 \leq f(t, v(t)) + \lambda v(0) - f(t, u(t)) - \lambda u(0) \leq \mu(v(t) - u(t))\ \forall\ u, v \in C[-r, 0]$ such that $u \leq v$ and for some λ, μ such that $0 < \mu < \lambda$.

In [16], PBVP for first order DDE $x'(t) = f(t, x, x_t)$, $t \in [0, 2\pi]$, $x(0) = x(2\pi)$, is investigated using contraction mapping theorem for operators whose domain and range are different Banach spaces. Monotone iterative method for PBVP of differential equation with piecewise constant argument (DEPCA) given by $x'(t) = f(t, x, x([t-1]))$, $t \in [0, T]$, $x(0) = x(T)$ is studied in [21]. Here $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function satisfying

$$f(t, u_1, v_1) - f(t, u_2, v_2) \geq -M_1(u_1 - u_2) - M_2(v_1 - v_2),$$

for $t \in [0, T], M_1, M_2 > 0$, $u_1, u_2, v_1, v_2 \in \mathbb{R}$ with $\phi(t) \leq u_2 \leq u_1 \leq \psi(t)$ and $\phi([t-1]) \leq v_2 \leq v_1 \leq \psi([t-1])$, where ϕ and ψ are lower and upper solutions of the given PBVP for DEPCA.

Monotone iterative method for anti-periodic BVP of first order ODE is developed in [7] and for BVP of first order DDE is developed in [11] and [16]. For recent work on monotone iterative method one may refer [10, 12, 14, 17, 19, 20].

In this paper, we discuss the following boundary value problem with linear condition (LBVP). Let $T > 0$, $\lambda > 0$, $I = [0, T]$, $0 < r < T$, $a, b \in \mathbb{R}$ such that $b < 0, a \neq 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. Then, the LBVP for FODDE, under investigation, is given by,

$$y'(t) + \lambda y(t) = f(t, y(t-r));\ t \in I, \tag{1.1}$$

$$y(t) = y(0);\ t \in [-r, 0], \tag{1.2}$$

$$ay(0) + by(T) = 0. \tag{1.3}$$

We first prove existence of solution between weakly coupled lower and upper solution for the problem. Next, we use this theorem on existence of solution to develope monotone iterative method for the LBVP of FODDE when $b < 0$.

Note that in many papers $f : I \times \mathbb{R} \to \mathbb{R}$ is taken to be continuous and Lipschitz or continuous and decreasing in second coordinate for establishing existence of a solution for initial and boundary value problems. In our paper we establish the existence of solution to (1.1)–(1.3) *for a function $f : I \times \mathbb{R} \to \mathbb{R}$ which is continuous and non-decreasing in the second coordinate.*

We use the following notations throughout the paper:

(1) $B = C[-r, T] \cap C'[0, T]$ is the set of all functions defined and continuous on $[-r, T]$ and which are continuously differentiable on $[0, T]$.

(2) $X = \{y \in B : y(t) = y(0),\ \forall\, t \in [-r, 0]\}$.

(3) $Y = \{y \in B : y(t) = 0,\ \forall\, t \in [-r, 0]\}$.

(4) $\|x\|_\infty = \sup_{t \in [-r,T]} |x(t)| + \sup_{t \in [0,T]} |x'(t)|,\ \forall x \in B$.

(5) $\|(y, \xi)\|_\star = \|y\|_\infty + |\xi|,\ \forall\, (y, \xi) \in Y \times \mathbb{R}$.

(6) For a linear operator $L : X \to Y \times \mathbb{R}$, let $\|L\| = \sup_{\|x\|_\infty = 1} \|Lx\|_\star$.

Also $(B, \|\cdot\|_\infty)$, $(X, \|\cdot\|_\infty)$, $(Y, \|\cdot\|_\infty)$ and $(Y \times \mathbb{R}, \|\cdot\|_\star)$ are Banach Spaces.

We now state Schauder's fixed point theorem given in [13].

Theorem 1.1 (Schauder's fixed point theorem)**.** *Let E be a normed linear space, $S : E \to E$ be a continuous and compact map such that $S(E)$ is bounded. Then, S has a fixed point.*

We need the following Lemmas which are similar to the Lemma 1.3 and Lemma 1.4 given in [1].

Lemma 1.2. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta e^{-\lambda T} \neq 0$. Let $L : X \to Y \times \mathbb{R}$ be an operator defined by $L[u(t)] = (z(t), \alpha y(0) + \beta y(T))$, where*

$$z(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ u(t) - u(0) + \lambda \displaystyle\int_0^t u(s)ds, & 0 \leq t \leq T. \end{cases}$$

Then

(1) *$L : X \to Y \times \mathbb{R}$ is a continuous linear operator.*

(2) *$L^{-1} : Y \times \mathbb{R} \to X$ exists and it is defined by*

$$L^{-1}(z(t), \gamma) = \begin{cases} \theta, & -r \leq t \leq 0 \\ \theta e^{-\lambda t} + \displaystyle\int_0^t e^{-\lambda(t-s)} z'(s)ds, & 0 \leq t \leq T. \end{cases}$$

where, $\theta = \dfrac{\gamma - \beta \displaystyle\int_0^T e^{-\lambda(T-s)} z'(s)ds}{\alpha + \beta e^{-\lambda T}}$.

(3) *$L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator.*

Lemma 1.3. *Let $F : I \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let $N : X \to Y \times \mathbb{R}$ be defined by $N[u(t)] = (z(t), g(u(0), u(T)))$, where*

$$z(t) = \begin{cases} 0, & -r \leq t \leq 0 \\ \displaystyle\int_0^t F(s, u(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Then $N : X \to Y \times \mathbb{R}$ is continuous and compact.

2. Existence theorem using Schauder's fixed point theorem.

In this section, we define weakly coupled lower and upper solutions for LBVP of FODDE to obtain existence of solution between them. We make use of Schauder's fixed point theorem to obtain our result.

First we have the following definitions.

Definition 2.1. *A function $\phi \in B$ is called a solution of LBVP* (1.1)–(1.3) *if*

1. $\phi'(t) + \lambda\phi(t) = f(t, \phi(t-r))$, $t \in I$,
2. $\phi(t) = \phi(0)$, $t \in [-r, 0]$,
3. $a\phi(0) + b\phi(T) = 0$.

Definition 2.2. *A pair of functions ϕ, $\psi \in B$ are called weakly coupled lower and upper solutions of LBVP* (1.1)–(1.3) *if*

1. $\phi'(t) + \lambda\phi(t) \leq f(t, \phi(t-r))$, $t \in I$,
2. $\psi'(t) + \lambda\psi(t) \geq f(t, \psi(t-r))$, $t \in I$,
3. $\phi(t) \leq \phi(0)$, $t \in [-r, 0]$,
4. $\psi(t) \geq \psi(0)$, $t \in [-r, 0]$,
5. $\phi(t) \leq \psi(t)$, $t \in [-r, T]$,
6. $\max\{a\phi(0) + b\phi(T), a\phi(0) + b\psi(T)\} \leq 0 \leq \min\{a\psi(0) + b\phi(T), a\psi(0) + b\psi(T)\}$.

Theorem 2.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. Let us suppose that for $x \leq y, f(t, x) \leq f(t, y)$, $\forall\ (t, x), (t, y) \in I \times \mathbb{R}$. Let ϕ, $\psi \in X$ be weakly coupled lower and upper solutions of LBVP given by* (1.1)–(1.3). *Then the LBVP* (1.1)–(1.3) *has atleast one solution $y \in X$ such that $\phi(t) \leq y(t) \leq \psi(t)$, $\forall\ t \in [-r, T]$.*

P r o o f. Since ϕ, $\psi \in X$ are weakly coupled lower and upper solutions of LBVP (1.1)–(1.3), we have

1. $\phi'(t) + \lambda\phi(t) \leq f(t, \phi(t-r))$, $t \in I$,
2. $\psi'(t) + \lambda\psi(t) \geq f(t, \psi(t-r))$, $t \in I$,
3. $\phi(t) \leq \phi(0)$, $t \in [-r, 0]$,
4. $\psi(t) \geq \psi(0)$, $t \in [-r, 0]$,
5. $\max\{a\phi(0) + b\phi(T), a\phi(0) + b\psi(T)\} \leq 0 \leq \min\{a\psi(0) + b\phi(T), a\psi(0) + b\psi(T)\}$.

Also as $b < 0$ we have, $\max\{a\phi(0)+b\phi(T), a\phi(0)+b\psi(T)\} = a\phi(0)+b\phi(T)$ and $\min\{a\psi(0) + b\phi(T), a\psi(0) + b\psi(T)\} = a\psi(0) + b\psi(T)$. Therefore $a\phi(0) + b\phi(T) \leq 0 \leq a\psi(0) + b\psi(T)$. Further, since ϕ, $\psi \in X, \phi(t) = \phi(0)$, $\forall\ t \in [-r, 0]$ and $\psi(t) = \psi(0)$, $\forall\ t \in [-r, 0]$.

Let $p : I \times \mathbb{R} \to \mathbb{R}$ be defined by $p(t, x) = \max\{\phi(t), \min\{x, \psi(t)\}\}$, $\forall\ (t, x) \in I \times \mathbb{R}$. Since, $\phi(t) \leq \max\{\phi(t), \{x, \psi(t)\}\}$, $\phi(t) \leq p(t, x)$, $\forall\ (t, x) \in I \times \mathbb{R}$. Also, $\phi(t) \leq \psi(t)$, $t \in [-r, T]$ implies $\phi(t) \leq \psi(t)$, $t \in I$. Further, $\min\{x, \psi(t)\} \leq \psi(t)$, $\forall\ (t, x) \in I \times \mathbb{R}$ implies that $\max\{\phi(t), \min\{x, \psi(t)\}\} \leq \psi(t)$, $\forall\ (t, x) \in$

$I \times \mathbb{R}$. Therefore $p(t,x) \leq \psi(t)\ \forall\ (t,x) \in I \times \mathbb{R}$. Thus $\phi(t) \leq p(t,x) \leq \psi(t)$ $\forall\ (t,x) \in I \times \mathbb{R}$. Hence $p : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined as $g(x,y) = ax + by\ \forall\ (x,y) \in \mathbb{R}^2$. Therefore $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and $g(y(0), y(T)) = ay(0) + by(T)$. Let $g^\star : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\begin{aligned} g^\star(x,y) &= p(0,x) - g[p(0,x), p(T,y)] \quad \forall\ (x,y) \in \mathbb{R}^2. \\ &= p(0,x) - ap(0,x) - bp(T,y) \quad \forall\ (x,y) \in \mathbb{R}^2. \\ &= (1-a)p(0,x) - bp(T,y) \quad \forall\ (x,y) \in \mathbb{R}^2. \end{aligned}$$

As $\alpha(0) \leq p(0,x) \leq \beta(0)\ \forall\ x \in \mathbb{R}$, we have $|p(0,x)| \leq K_1 = \max\{|\alpha(0)|+1, |\beta(0)|\}$ $\forall\ x \in \mathbb{R}$. Also $\alpha(T) \leq p(T,y) \leq \beta(T)\ \forall\ y \in \mathbb{R}$ implies that $|p(T,y)| \leq K_2 = \max\{|\alpha(T)|+1, |\beta(T)|\}\ \forall\ y \in \mathbb{R}$. Therefore,

$$\begin{aligned} |g^\star(x,y)| &\leq |1-a||p(0,x)| + |b||p(T,y)| \quad \forall\ (x,y) \in \mathbb{R}^2. \\ &\leq |1-a|K_1 + |b|K_2 \quad \forall\ (x,y) \in \mathbb{R}^2. \end{aligned}$$

Thus $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is bounded. Also $g^\star : \mathbb{R}^2 \to \mathbb{R}$ is continuous, as p and g are continuous.

Let $F^\star : I \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F^\star(t,y) = \begin{cases} f(t, \phi(t-r)), & y \leq \phi(t-r). \\ f(t,y), & \phi(t-r) \leq y \leq \psi(t-r). \\ f(t, \psi(t-r)), & y \geq \psi(t-r). \end{cases}$$

Then $F^\star : I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $L : X \to Y \times \mathbb{R}$ be defined by $L[y(t)] = (u(t), y(0))$, where

$$u(t) = \begin{cases} 0, & -r \leq t \leq 0. \\ y(t) - y(0) + \lambda \displaystyle\int_0^t y(s)ds, & 0 \leq t \leq T. \end{cases}$$

By Lemma 1.2, $L : X \to Y \times \mathbb{R}$ is linear, bijective and continuous. Hence $L^{-1} : Y \times \mathbb{R} \to X$ exists and is given by

$$L^{-1}(u(t), \gamma) = \begin{cases} \gamma, & -r \leq t \leq 0 \\ \gamma e^{-\lambda t} + \displaystyle\int_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \leq t \leq T. \end{cases}$$

Also by bounded inverse theorem, $L^{-1} : Y \times \mathbb{R} \to X$ is bounded and hence continuous.

Let $N : X \to Y \times \mathbb{R}$ be defined as $N[y(t)] = (v(t), g^\star(y(0), y(T)))$, where

$$v(t) = \begin{cases} 0, & -r \le t \le 0 \\ \displaystyle\int_0^t F^\star(s, y(s-r))ds, & 0 \le t \le T. \end{cases}$$

By Lemma 1.3, $N : X \to Y \times \mathbb{R}$ is compact and continuous.

Consider $S = L^{-1}N : X \to X$. Then

$$S[y(t)] = \begin{cases} g^\star(y(0), y(T))), & -r \le t \le 0 \\ g^\star(y(0), y(T)))e^{-\lambda t} + \displaystyle\int_0^t e^{-\lambda(t-s)} F^\star(s, y(s-r))ds, & 0 \le t \le T. \end{cases}$$

A fixed point of $S : X \to X$ is a solution of

$$y'(t) + \lambda y(t) = F^\star(t, y(t-r)), \quad t \in I, \tag{2.1}$$

$$y(t) = y(0), \quad t \in [-r, 0],$$

$$y(0) = g^\star(y(0), y(T))). \tag{2.2}$$

Since $L^{-1} : Y \times \mathbb{R} \to X$ is continuous linear operator and $N : X \to Y \times \mathbb{R}$ is compact, $S : X \to X$ is a compact operator. Further $S : X \to X$ is continuous being composition of continuous operators. As p and $F^\star$ are bounded, we have $S(X)$ is a bounded subset of $(X, \|\cdot\|_\infty)$. Therefore by Schauder's fixed point theorem, $S : X \to X$ has a fixed point $y \in X$. This implies that y is a solution of (2.1), (1.2) and (2.2).

We show that $\phi(t) \le y(t) \le \psi(t)$, $\forall\ t \in [-r, T]$. We first show that $\phi(0) \le y(0) \le \psi(0)$. Let if possible, $y(0) > \psi(0)$. Therefore

$$p(0, y(0)) = \max\{\phi(0), \min\{y(0), \psi(0)\}\} \ = \max\{\phi(0), \psi(0)\} = \psi(0).$$

As $p(T, y(T)) \le \psi(T)$ we have, $b\psi(T) \le bp(T, y(T))$. This implies

$$0 \le a\psi(0) + b\psi(T) \le a\psi(0) + bp(T, y(T)).$$

Therefore

$$\begin{aligned} y(0) &= g^\star(y(0), y(T)) \\ &= p(0, y(0)) - ap(0, y(0)) - bp(T, y(T)) \\ &= \psi(0) - a\psi(0) - bp(T, y(T)) \end{aligned}$$

$$\begin{aligned} &= \psi(0) - [a\psi(0) + bp(T, y(T))] \\ &\leq \psi(0), \end{aligned}$$

which is a contradiction. Hence we have $y(0) \leq \psi(0)$.

Similarly it can be proved that $\phi(0) \leq y(0)$. Therefore $\phi(0) \leq y(0) \leq \psi(0)$. This implies $\phi(t) \leq y(t) \leq \psi(t)$, $\forall\ t \in [-r, 0]$.

The remaining proof follows from the proof of Theorem 3.3 in [1]. □

Consider the following example of LBVP to illustrate Theorem 2.1.

$$\begin{aligned} &w'(t) + w(t) = (w(t-1) + 1)e^{-t}, \quad t \in [0, 1], \\ &w(t) = w(0), \quad t \in [-1, 0], \\ &w(0) - w(1) = 0. \end{aligned}$$

Here $\lambda = 1$, $a = 1$, $b = -1$, $r = 1$, $T = 1$, $I = [0, 1]$ and $f(t, u) = (u + 1)e^{-t}$, $\forall\ (t, u) \in I \times \mathbb{R}$. Therefore $f(t, w(t-1)) = (w(t-1) + 1)e^{-t}$. Also f is non-decreasing in the second coordinate. Let $\phi : [-1, 1] \to \mathbb{R}$ be defined as

$$\phi(t) = \begin{cases} \dfrac{1 - e^{-1}}{e - 2}, & -1 \leq t \leq 0, \\ \dfrac{e^{-t}}{e - 2} + \left(\dfrac{e - 1}{e - 2}\right) te^{-t} - \dfrac{e^{-1}}{e - 2}, & 0 \leq t \leq 1. \end{cases}$$

Then $\phi(0) = \dfrac{1 - e^{-1}}{e - 2} = \phi(1)$. This gives, $\phi(0) - \phi(1) = 0$. Let $0 \leq t \leq 1$. This implies $-1 \leq t - 1 \leq 0$. Therefore $\phi(t - 1) = \phi(0)$. So we have

$$\phi'(t) + \phi(t) = \left(\frac{e - 1}{e - 2}\right) e^{-t} - \frac{e^{-1}}{e - 2}$$

and

$$(\phi(t - 1) + 1)e^{-t} = \left(\frac{e - 1}{e - 2}\right) e^{-t} - \frac{e^{-1}e^{-t}}{e - 2}.$$

Thus $\phi'(t) + \phi(t) \leq (\phi(t - 1) + 1)e^{-t}$.

Let $\psi : [-1, 1] \to \mathbb{R}$ be defined as

$$\psi(t) = \begin{cases} \dfrac{1 + e^{-1}}{e - 2}, & -1 \leq t \leq 0, \\ \dfrac{e^{-t}}{e - 2} + \left(\dfrac{e - 1}{e - 2}\right) te^{-t} + \dfrac{e^{-1}}{e - 2}, & 0 \leq t \leq 1. \end{cases}$$

Then $\psi(0) = \dfrac{1+e^{-1}}{e-2} = \psi(1)$. This gives $\psi(0) - \psi(1) = 0$. Let $0 \le t \le 1$. This implies $-1 \le t-1 \le 0$. Therefore $\psi(t-1) = \psi(0)$. So we have

$$\psi'(t) + \psi(t) = \left(\frac{e-1}{e-2}\right) e^{-t} + \frac{e^{-1}}{e-2}$$

and

$$(\psi(t-1)+1)e^{-t} = \left(\frac{e-1}{e-2}\right) e^{-t} + \frac{e^{-1}e^{-t}}{e-2}.$$

Thus $\psi'(t) + \psi(t) \ge (\psi(t-1)+1)e^{-t}$. Therefore we have

1. $\phi'(t) + \phi(t) \le (\phi(t-1)+1)e^{-t}$, $t \in [0,1]$,
2. $\psi'(t) + \psi(t) \ge (\psi(t-1)+1)e^{-t}$, $t \in [0,1]$,
3. $\phi(t) = \phi(0)$, $t \in [-1,0]$,
4. $\psi(t) = \psi(0)$, $t \in [-1,0]$,
5. $\phi(t) \le \psi(t)$, $t \in [-1,1]$,
6. $\phi(0) - \phi(1) = 0 = \psi(0) - \psi(1)$.

So we can say that ϕ and ψ are coupled lower and upper solutions of given LBVP.

Now let $y : [-1,1] \to \mathbb{R}$ be defined as

$$y(t) = \begin{cases} \dfrac{1}{e-2}, & -1 \le t \le 0, \\ \dfrac{e^{-t}}{e-2} + \left(\dfrac{e-1}{e-2}\right) te^{-t}, & 0 \le t \le 1. \end{cases}$$

Then y is a solution of given LBVP such that $\phi \le y \le \psi$ on $[-1,1]$.

3. Monotone iterative method. In this section we develope Monotone Iterative Method for LBVP (1.1)–(1.3), by defining monotonic sequences of functions which are solutions of some LBVP of linear first order DDE from given weakly coupled lower and upper solutions of (1.1)–(1.3).

Theorem 3.1. *Let $f : I \times \mathbb{R} \to \mathbb{R}$ be continuous. Let us suppose that for $y \le x, f(t,x) - f(t,y) \ge \mu(x-y)$, $\forall\ (t,x),(t,y) \in I \times \mathbb{R}$ and for some $\mu > 0$. Let ϕ_0, $\psi_0 \in X$ be weakly coupled lower and upper solutions of LBVP (1.1)–(1.3). Then, there are sequences of monotonic functions (ϕ_n) and (ψ_n) converging in $(X, \|.\|_\infty)$ such that $\lim\limits_{n\to\infty} \phi_n$ and $\lim\limits_{n\to\infty} \psi_n$ are solutions of (1.1)–(1.3).*

P r o o f. Since ϕ_0, $\psi_0 \in X$ are weakly coupled lower and upper solutions of LBVP (1.1)–(1.3), we have

1. $\phi_0'(t) + \lambda\phi_0(t) \leq f(t, \phi_0(t-r))$, $t \in I$,

2. $\psi_0'(t) + \lambda\psi_0(t) \geq f(t, \psi_0(t-r))$, $t \in I$,

3. $\phi_0(t) \leq \phi_0(0)$, $t \in [-r, 0]$,

4. $\psi_0(t) \geq \psi_0(0)$, $t \in [-r, 0]$,

5. $\max\{a\phi_0(0) + b\phi_0(T), a\phi_0(0) + b\psi_0(T)\} \leq 0 \leq \min\{a\psi_0(0) + b\phi_0(T), a\psi_0(0) + b\psi_0(T)\}$.

Now ϕ_0, $\psi_0 \in X$ implies that $\phi_0(t) = \phi_0(0)$, $\forall\ t \in [-r, 0]$ and $\psi_0(t) = \psi_0(0)$, $\forall\ t \in [-r, 0]$. Also as $b < 0$ we have

$$\max\{a\phi_0(0) + b\phi_0(T), a\phi_0(0) + b\psi_0(T)\} = a\phi_0(0) + b\phi_0(T)$$

and

$$\min\{a\psi_0(0) + b\phi_0(T), a\psi_0(0) + b\psi_0(T)\} = a\psi_0(0) + b\psi_0(T).$$

Therefore $a\phi_0(0) + b\phi_0(T) \leq 0 \leq a\psi_0(0) + b\psi_0(T)$. Consider

$$y'(t) + \lambda y(t) = f(t, \phi_0(t-r)) + \mu[y(t-r) - \phi_0(t-r)], \quad t \in I, \tag{3.1}$$

$$y(t) = y(0), \quad t \in [-r, 0], \tag{3.2}$$

$$ay(0) + by(T) = 0. \tag{3.3}$$

Let $g_{\phi_0}(t, u) = f(t, \phi_0(t-r)) + \mu[u - \phi_0(t-r)]$, $\forall\ (t, u) \in I \times \mathbb{R}$. This implies $g_{\phi_0}(t, x(t-r)) = f(t, \phi_0(t-r)) + \mu[x(t-r) - \phi_0(t-r)]$, $\forall\ t \in I$. Let $(t, x), (t, y) \in I \times \mathbb{R}$ such that $x \leq y$. Therefore $x - \phi_0(t-r) \leq y - \phi_0(t-r)$. This gives $\mu[x - \phi_0(t-r)] \leq \mu[y - \phi_0(t-r)]$. Therefore $f(t, \phi_0(t-r)) + \mu[x - \phi_0(t-r)] \leq f(t, \phi_0(t-r)) + \mu[y - \phi_0(t-r)]$. Thus $g_{\phi_0}(t, x) \leq g_{\phi_0}(t, y)$, $\forall\ (t, x), (t, y) \in I \times \mathbb{R}$ such that $x \leq y$. Also

$$\begin{aligned} \phi_0'(t) + \lambda\phi_0(t) &\leq f(t, \phi_0(t-r)) \\ &= f(t, \phi_0(t-r)) + \mu[\phi_0(t-r) - \phi_0(t-r)] \\ &= g_{\phi_0}(t, \phi_0(t-r)). \end{aligned}$$

Further

$$\begin{aligned} \psi_0'(t) + \lambda\psi_0(t) &\geq f(t, \psi_0(t-r)) \\ &\geq f(t, \phi_0(t-r)) + \mu[\psi_0(t-r) - \phi_0(t-r)] \\ &= g_{\phi_0}(t, \psi_0(t-r)). \end{aligned}$$

We already know that

1. $\phi_0(t) = \phi_0(0), \ \ t \in [-r, 0]$,
2. $\psi_0(t) = \psi_0(0), \ \ t \in [-r, 0]$,
3. $\phi_0(t) \leq \psi_0(t), \ \ t \in [-r, T]$,
4. $a\phi_0(0) + b\phi_0(T) \leq 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_0 and ψ_0 are weakly coupled lower and upper solutions of (3.1)–(3.3). Then by Theorem 2.1 LBVP (3.1)–(3.3) has a solution $\phi_1 \in X$ such that $\phi_0 \leq \phi_1 \leq \psi_0$ on $[-r, T]$. Therefore

$$\begin{aligned}
&\phi_1'(t) + \lambda\phi_1(t) = f(t, \phi_0(t-r)) + \mu[\phi_1(t-r) - \phi_0(t-r)], \quad t \in I.\\
&\phi_1(t) = \phi_1(0), \quad t \in [-r, 0].\\
&a\phi_1(0) + b\phi_1(T) = 0.
\end{aligned}$$

Thus $\phi_1'(t) + \lambda\phi_1(t) = f(t, \phi_0(t-r)) + \mu[\phi_1(t-r) - \phi_0(t-r)] \leq f(t, \phi_1(t-r))$. Also

1. $\psi_0'(t) + \lambda\psi_0(t) \geq f(t, \psi_0(t-r))$, $t \in I$,
2. $\psi_0(t) = \psi_0(0)$, $t \in [-r, 0]$,
3. $\phi_1(t) = \phi_1(0)$, $t \in [-r, 0]$,
4. $\phi_1(t) \leq \psi_0(t)$, $t \in [-r, T]$,
5. $a\phi_1(0) + b\phi_1(T) = 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_1 and $\psi_0 \in X$ are weakly coupled lower and upper solutions of LBVP (1.1)–(1.3). Consider

$$\begin{aligned}
&y'(t) + \lambda y(t) = f(t, \phi_1(t-r)) + \mu[y(t-r) - \phi_1(t-r)], \quad t \in I,\\
&y(t) = y(0), \ t \in [-r, 0],\\
&ay(0) + by(T) = 0.
\end{aligned} \tag{3.4}$$

Let $g_{\phi_1}(t, u) = f(t, \phi_1(t-r)) + \mu[u - \phi_1(t-r)]$, $\forall\ (t, u) \in I \times \mathbb{R}$. Therefore $g_{\phi_1}(t, x(t-r)) = f(t, \phi_1(t-r)) + \mu[x(t-r) - \phi_1(t-r)]$, $\forall\ t \in I$. Let $(t, x), (t, y) \in I \times \mathbb{R}$ such that $x \leq y$. This gives $x - \phi_1(t-r) \leq y - \phi_1(t-r)$. Therefore $\mu[x - \phi_1(t-r)] \leq \mu[y - \phi_1(t-r)]$. This implies $f(t, \phi_1(t-r)) + \mu[x - \phi_1(t-r)] \leq f(t, \phi_1(t-r)) + \mu[y - \phi_1(t-r)]$. Therefore $g_{\phi_1}(t, x) \leq g_{\phi_1}(t, y)$, $\forall\ (t, x), (t, y) \in I \times \mathbb{R}$ such that $x \leq y$. Also

$$\begin{aligned}
\phi_1'(t) + \lambda\phi_1(t) &\leq f(t, \phi_1(t-r))\\
&= f(t, \phi_1(t-r)) + \mu[\phi_1(t-r) - \phi_1(t-r)]
\end{aligned}$$

$$= g_{\phi_1}(t, \phi_1(t-r)).$$

Further

$$\begin{aligned}\psi_0'(t) + \lambda\psi_0(t) &\geq f(t, \psi_0(t-r)) \\ &\geq f(t, \phi_1(t-r)) + \mu[\psi_0(t-r) - \phi_1(t-r)] \\ &= g_{\phi_1}(t, \psi_0(t-r)).\end{aligned}$$

We already know that

1. $\phi_1(t) = \phi_1(0)$, $t \in [-r, 0]$,
2. $\psi_0(t) = \psi_0(0)$, $t \in [-r, 0]$,
3. $\phi_1(t) \leq \psi_0(t)$, $t \in [-r, T]$,
4. $a\phi_1(0) + b\phi_1(T) = 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_1 and ψ_0 are weakly coupled lower and upper solutions of (3.4), (3.2) and (3.3). This implies that LBVP (3.4), (3.2) and (3.3) has a solution $\phi_2 \in X$ such that $\phi_1 \leq \phi_2 \leq \psi_0$ on $[-r, T]$ (By Theorem 2.1). Therefore

$$\begin{aligned}&\phi_2'(t) + \lambda\phi_2(t) = f(t, \phi_1(t-r)) + \mu[\phi_2(t-r) - \phi_1(t-r)], \quad t \in I. \\ &\phi_2(t) = \phi_2(0), \quad t \in [-r, 0]. \\ &a\phi_2(0) + b\phi_2(T) = 0.\end{aligned}$$

Thus $\phi_2'(t) + \lambda\phi_2(t) = f(t, \phi_1(t-r)) + \mu[\phi_2(t-r) - \phi_1(t-r)] \leq f(t, \phi_2(t-r))$. Also

1. $\psi_0'(t) + \lambda\psi_0(t) \geq f(t, \psi_0(t-r))$, $t \in I$,
2. $\psi_0(t) = \psi_0(0)$, $t \in [-r, 0]$,
3. $\phi_2(t) = \phi_2(0)$, $t \in [-r, 0]$,
4. $\phi_2(t) \leq \psi_0(t)$, $t \in [-r, T]$,
5. $a\phi_2(0) + b\phi_2(T) = 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_2 and $\psi_0 \in X$ are weakly coupled lower and upper solutions of LBVP for FODDE (1.1)–(1.3).

Continuing in this manner, let $\phi_n \in X$ be a solution of

$$\begin{aligned}&y'(t) + \lambda y(t) = f(t, \phi_{n-1}(t-r)) + \mu[y(t-r) - \phi_{n-1}(t-r)], \quad t \in I, \\ &y(t) = y(0), \quad t \in [-r, 0], \\ &ay(0) + by(T) = 0, \text{ such that } \phi_{n-1} \leq \phi_n \leq \psi_0 \text{ on } [-r, T].\end{aligned} \tag{3.5}$$

Consider

$$\begin{aligned}(3.6)\qquad & y'(t)+\lambda y(t)=f(t,\phi_n(t-r))+\mu[y(t-r)-\phi_n(t-r)], \quad t\in I,\\ & y(t)=y(0),\ t\in[-r,0],\\ & ay(0)+by(T)=0.\end{aligned}$$

Let $g_{\phi_n}(t,u)=f(t,\phi_n(t-r))+\mu[u-\phi_n(t-r)]$, $\forall\ (t,u)\in I\times\mathbb{R}$. Therefore

$$g_{\phi_n}(t,x(t-r))=f(t,\phi_n(t-r))+\mu[x(t-r)-\phi_n(t-r)], \quad \forall\ t\in I$$

and

$$g_{\phi_n}(t,x)\le g_{\phi_n}(t,y), \quad \forall\ (t,x),(t,y)\in I\times\mathbb{R}$$

such that $x\le y$. Also ϕ_n and ψ_0 are weakly coupled lower and upper solutions of (3.6), (3.2) and (3.3). Thus LBVP (3.6), (3.2) and (3.3) has a solution $\phi_{n+1}\in X$ such that $\phi_n\le\phi_{n+1}\le\psi_0$ on $[-r,T]$ (By Theorem 2.1). Therefore

$$\begin{aligned}&\phi'_{n+1}(t)+\lambda\phi_{n+1}(t)=f(t,\phi_n(t-r))+\lambda[\phi_{n+1}(t-r)-\phi_n(t-r)], \quad t\in I,\\ &\phi_{n+1}(t)=\phi_{n+1}(0),\ t\in[-r,0],\\ &a\phi_{n+1}(0)+b\phi_{n+1}(T)=0.\end{aligned}$$

Thus, we get a monotonic increasing sequence of functions $(\phi_n)\subseteq X$ such that $\phi_n\le\psi_0$, $\forall\ n\in\mathbb{N}$ on $[-r,T]$ and

$$\begin{aligned}&\phi'_{n+1}(t)+\lambda\phi_{n+1}(t)=f(t,\phi_n(t-r))+\mu[\phi_{n+1}(t-r)-\phi_n(t-r)], \quad t\in I,\\ &\phi_{n+1}(t)=\phi_{n+1}(0), \quad t\in[-r,0],\\ &a\phi_{n+1}(0)+b\phi_{n+1}(T)=0.\end{aligned}$$

Now for each fixed $t\in[-r,T]$, $\{\phi_n(t)\}$ is a monotonic increasing sequence in $\mathbb{R}$ which is bounded above by $\psi_0(t)$. Therefore, by monotone convergence theorem for sequences in $\mathbb{R}$, $\{\phi_n(t)\}$ is convergent in $\mathbb{R}$ for each $t\in[-r,T]$. Thus (ϕ_n) is pointwise convergent on $[-r,T]$. Let $\lim\limits_{n\to\infty}\phi_n(t)=\phi(t)$, $\forall\ t\in[-r,T]$. Since

$$\phi'_{n+1}(t)+\lambda\phi_{n+1}(t)=f(t,\phi_n(t-r))+\mu[\phi_{n+1}(t-r)-\phi_n(t-r)],$$

we have

$$\phi_{n+1}(t)e^{\lambda t}-\phi_{n+1}(0)=\int_0^t e^{\lambda s}\{f(s,\phi_n(s-r))+\mu[\phi_{n+1}(s-r)-\phi_n(s-r)]\}ds.$$

This implies

$$\phi_{n+1}(t) = \phi_{n+1}(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s, \phi_n(s-r))ds$$
$$+ \int_0^t e^{-\lambda(t-s)} \{\mu[\phi_{n+1}(s-r) - \phi_n(s-r)]\}ds.$$

Now taking limit as $n \to \infty$ on both sides of the above equality we get

$$\lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \phi_{n+1}(0)e^{-\lambda t} + \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} f(s, \phi_n(s-r))ds$$
$$+ \lim_{n\to\infty} \int_0^t e^{-\lambda(t-s)} \{\mu[\phi_{n+1}(s-r) - \phi_n(s-r)]\}ds.$$

Therefore, by Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} \phi_{n+1}(t) = \lim_{n\to\infty} \phi_{n+1}(0)e^{-\lambda t} + \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} f(s, \phi_n(s-r))ds$$
$$+ \int_0^t \lim_{n\to\infty} e^{-\lambda(t-s)} \{\mu[\phi_{n+1}(s-r) - \phi_n(s-r)]\}ds.$$

Therefore $\phi(t) = \phi(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)} f(s, \phi(s-r))ds$.

Also $\phi(t) = \phi(0)$, $\forall\ t \in [-r, 0]$ and $a\phi(0) + b\phi(T) = 0$. Thus ϕ is a solution of (1.1)–(1.3). As ϕ is continuous, $\{\phi_n\}$ is increasing and $\lim_{n\to\infty} \phi_n(t) = \phi(t)$, we have by Dini's Theorem for uniform convergence of sequence of functions, $\phi_n \to \phi$ uniformly on I. Therefore

$$\lim_{n\to\infty} \phi'_{n+1}(t) = -\lambda \lim_{n\to\infty} \phi_{n+1}(t)$$
$$+ \lim_{n\to\infty} f(t, \phi_n(t-r)) + \mu[\phi_{n+1}(t-r) - \phi_n(t-r)].$$
$$= -\lambda\phi(t) + f(t, \phi(t-r)).$$

Since the left hand side of the above limit is the uniform limit of sequence of functions, we have $\{\phi_n'\}$ converging uniformly on I. Now we have $\phi_n \to \phi$ pointwise on $[-r, T]$ and $\phi_n \to \phi$ uniformly on I. Also $\phi_n(t) = \phi_n(0)$, $\forall\ t \in [-r, 0]$ and $\forall\ n \in \mathbb{N}$, $\phi(t) = \phi(0)$, $\forall\ t \in [-r, 0]$. Therefore we have $\phi_n \to \phi$ uniformly on $[-r, T]$. This implies that ϕ is continuous on $[-r, T]$. Also ϕ is differentiable on I and ϕ' is continuous on I. Thus $\phi \in X$. As $\phi_n \to \phi$ uniformly on $[-r, T]$ and $\phi_n' \to \phi'$ uniformly on I, we have $\phi_n \to \phi$ on $(X, \|.\|_\infty)$ and ϕ is a solution of LBVP (1.1)–(1.3).

Next consider

$$y'(t) + \lambda y(t) = f(t, \psi_0(t-r)) + \mu[y(t-r) - \psi_0(t-r)], \quad t \in I, \tag{3.7}$$

$$y(t) = y(0), \quad t \in [-r, 0], \tag{3.8}$$

$$ay(0) + by(T) = 0. \tag{3.9}$$

Let $h_{\psi_0}(t, u) = f(t, \psi_0(t-r)) + \mu[u - \psi_0(t-r)]$, $\forall\ (t, u) \in I \times \mathbb{R}$. Therefore $h_{\psi_0}(t, x(t-r)) = f(t, \psi_0(t-r)) + \mu[x(t-r) - \psi_0(t-r)]$, $\forall\ t \in I$ and $h_{\psi_0}(t, x) \leq h_{\psi_0}(t, y)$, $\forall\ (t, x), (t, y) \in I \times \mathbb{R}$ such that $x \leq y$. Also

$$\begin{aligned} \phi_0'(t) + \lambda\phi_0(t) &\leq f(t, \phi_0(t-r)) \\ &\leq f(t, \psi_0(t-r)) + \mu[\phi_0(t-r) - \psi_0(t-r)] \\ &= h_{\psi_0}(t, \phi_0(t-r)). \end{aligned}$$

Further

$$\begin{aligned} \psi_0'(t) + \lambda\psi_0(t) &\geq f(t, \psi_0(t-r)) \\ &= f(t, \psi_0(t-r)) + \mu[\psi_0(t-r) - \psi_0(t-r)] \\ &= h_{\psi_0}(t, \psi_0(t-r)). \end{aligned}$$

We already know that

1. $\phi_0(t) = \phi_0(0)$, $t \in [-r, 0]$,
2. $\psi_0(t) = \psi_0(0)$, $t \in [-r, 0]$,
3. $\phi_0(t) \leq \psi_0(t)$, $t \in [-r, T]$,
4. $a\phi_0(0) + b\phi_0(T) \leq 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_0 and ψ_0 are weakly coupled lower and upper solutions of (3.1)–(3.3). Then by Theorem 2.1 LBVP (3.7)–(3.9) has a solution $\psi_1 \in X$ such that $\phi_0 \leq \psi_1 \leq \psi_0$ on $[-r, T]$. Therefore

$$\psi_1'(t) + \lambda\psi_1(t) = f(t, \psi_0(t-r)) + \mu[\psi_1(t-r) - \psi_0(t-r)], \quad t \in I,$$
$$\psi_1(t) = \psi_1(0), \quad t \in [-r, 0],$$
$$a\psi_1(0) + b\psi_1(T) = 0.$$

Thus $\psi_1'(t) + \lambda\psi_1(t) \geq f(t, \psi_1(t-r))$, $t \in I$. Also

1. $\psi_0'(t) + \lambda\psi_0(t) \geq f(t, \psi_0(t-r)), \ \ t \in I,$
2. $\psi_0(t) = \psi_0(0)$, $t \in [-r, 0]$,
3. $\phi_1(t) = \phi_1(0)$, $t \in [-r, 0]$,
4. $\phi_1(t) \leq \psi_0(t)$, $t \in [-r, T]$,
5. $a\phi_1(0) + b\phi_1(T) = 0 \leq a\psi_0(0) + b\psi_0(T)$.

Therefore ϕ_0 and $\psi_1 \in X$ are weakly coupled lower and upper solutions of LBVP (1.1)–(1.3).

Continuing in this manner, we get a monotonic decreasing sequence of functions $(\psi_n) \subseteq X$ such that $\phi_0 \leq \psi_n$, $\forall\ n \in \mathbb{N}$ on $[-r, T]$ and

$$\psi_{n+1}'(t) + \lambda\psi_{n+1}(t) = f(t, \psi_n(t-r)) + \mu[\psi_{n+1}(t-r) - \psi_n(t-r)], \quad t \in I,$$
$$\psi_{n+1}(t) = \psi_{n+1}(0), \quad t \in [-r, 0],$$
$$a\psi_{n+1}(0) + b\psi_{n+1}(T) = 0.$$

Further, $\{\psi_n\}$ converges in $(X, \|\cdot\|_\infty)$ and $\lim\limits_{n\to\infty} \psi_n$ is a solution of LBVP (1.1)–(1.3) □

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