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APPROXIMATION OF DOUBLE FOURIER SERIES FOR FUNCTIONS IN $H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$ CLASS USING SUB-MATRIX MEANS

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ABSTRACT. Here, we have expanded the outcome related to the approximation of double Fourier series using sub-matrix means, encompassing a broader set of sequences and a more inclusive class of functions $H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$ within a mixed Lebesgue space.

1. Introduction. Let $f \in L_{\vec{p}}[\mathbb{T}^2]$ be a periodic function with period 2π in each variable, where $\mathbb{T} = [-\pi, \pi]$. Then, the double Fourier series for the function f is defined as

$$\begin{aligned} & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} [\alpha_{mn} \cos(mz) \cos(nw) + \beta_{mn} \sin(mz) \cos(nw) \\ (1.1) \quad & + \gamma_{mn} \cos(mz) \sin(nw) + \delta_{mn} \sin(mz) \sin(nw)] \\ & = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda_{mn} A_{mn}(z, w), \end{aligned}$$

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where,

$$(1.2) \quad \alpha_{mn} = \frac{1}{\pi^2} \int \int_{\mathbb{T}^2} f(u, v) \cos(mu) \cos(nv) dudv,$$

$$(1.3) \quad \beta_{mn} = \frac{1}{\pi^2} \int \int_{\mathbb{T}^2} f(u, v) \sin(mu) \cos(nv) dudv,$$

$$(1.4) \quad \gamma_{mn} = \frac{1}{\pi^2} \int \int_{\mathbb{T}^2} f(u, v) \cos(mu) \sin(nv) dudv,$$

$$(1.5) \quad \delta_{mn} = \frac{1}{\pi^2} \int \int_{\mathbb{T}^2} f(u, v) \sin(mu) \sin(nv) dudv,$$

are the Fourier coefficients of the function $f(z, w)$, for $m, n = 0, 1, 2, 3 \dots$ and $\lambda_{m,n} = 1, \lambda_{m,0} = \lambda_{0,n} = \frac{1}{2}$, for $m, n \geq 1, \lambda_{0,0} = \frac{1}{4}$.

Let $\{\mathcal{S}_{mn}\}$ be the sequence of partial sums of double infinite series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}$ and the sub-matrix means of the sequence $\{\mathcal{S}_{mn}\}$ is given by

$$\mathcal{T}_{mn} = \sum_{i=0}^m \sum_{j=0}^n a_{m,i} b_{n,j} \mathcal{S}_{mn}, \quad m, n \geq 0,$$

where $A = (a_{m,i})$ and $B = (b_{n,j})$ are two lower triangular infinite matrices such that

$$(1.6) \quad a_{m,i} = \begin{cases} \geq 0, & \text{if } i \leq m; \\ 0, & \text{if } i > m. \end{cases} \quad (i, m = 0, 1, 2, \dots) \text{ and } \sum_{i=1}^m a_{m,i} = 1,$$

$$(1.7) \quad b_{n,j} = \begin{cases} \geq 0, & \text{if } j \leq n; \\ 0, & \text{if } j > n. \end{cases} \quad (j, n = 0, 1, 2, \dots) \text{ and } \sum_{j=1}^n b_{n,j} = 1.$$

The double series $\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}$ is said to be (A, B) summable to a finite number s , if $\mathcal{T}_{mn} \rightarrow s$ as $m, n \rightarrow \infty$. The conditions of regularity for double matrix summability are given by

$$\sum_{i=0}^m \sum_{j=0}^n a_{m,i} b_{n,j} \rightarrow 1 \text{ as } m, n \rightarrow \infty$$

$$\lim_{m,n} \sum_{j=0}^n |a_{m,i} b_{n,j}| = 0 \text{ for each } i = 1, 2, \dots$$

$$\lim_{m,n} \sum_{i=0}^m |a_{m,i} b_{n,j}| = 0 \text{ for each } j = 1, 2, \dots$$

In the realm of sequence analysis, a sequence $c = \{c_n\}$ comprising nonnegative numbers that approach zero is designated as having rest bounded variation, denoted as $c \in RBVS$, if it has the property $\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c)c_m$ for all natural numbers m , where $K(c)$ is a constant solely depending on c .

Let $C_{2\pi}$ be the space of all continuous 2π -periodic functions on $\mathbb{T}^2$. For $0 < \alpha \leq 1$, $0 < \beta \leq 1$, define $H_{\alpha,\beta}$ as

$$H_{\alpha,\beta} = \{f \in C_{2\pi} : \forall z, w |f(z+t, w+s) - f(z, w)| = O(|t|^\alpha + |s|^\beta)\},$$

then $(H_{\alpha,\beta}, \|\cdot\|_{\alpha,\beta})$ is a Banach space, under the norm

$$\|f\|_{\alpha,\beta} = \|f\|_{\infty} + \sup_{t \neq 0, s \neq 0} \frac{|f(z+t, w+s) - f(z, w)|}{|t|^\alpha + |s|^\beta}.$$

In a series of studies, such as [5] and [4, 1], the degree of approximation for functions within the $Lip(\alpha, \beta)$ and $H_{\alpha,\beta}$ classes respectively, was investigated. In [3], degrees of the approximations by some special matrix means of conjugate Fourier series, was studied

Trigonometric polynomials of the form

$$(1.8) \quad S_{m,n}^{\lambda}(f; z, w) = \sum_{i=0}^{\lambda(m)} \sum_{j=0}^{\lambda(n)} a_{\lambda(m),i} b_{\lambda(n),j} s_{i,j}(f; z, w),$$

were utilized for this purpose, where $\{a_{\lambda(m),i}\} \in RBVS$ and $\{b_{\lambda(n),j}\} \in RBVS$ with respect to $0 \leq i \leq m$ and $0 \leq j \leq n$, satisfying specified conditions (1.6) and (1.7) respectively. Moreover, strict increasing sequences $\lambda(m)$ and $\lambda(n)$ played a crucial role in their approach.

In the work by the author in [7], a generalized class, denoted $\gamma RBVS$, is introduced. For a fixed sequence of positive numbers $\gamma = \{\gamma_n\}$ a nonnegative null-sequence $c = \{c_n\}$ belongs to the class $\gamma RBVS$ if

$$\sum_{n=m}^{\infty} |c_n - c_{n+1}| \leq K(c, \gamma) \gamma_m$$

holds for all $m \in N$. This is expressed as $c \in \gamma RBVS$ indicating that c is a γ rest bounded variation sequence. Notably, the $\gamma RBVS$ class is more inclusive than the $RBVS$ class, converging to identity only when $\gamma_n = Kc_n$, for some positive number K . Using [7, Lemma 3.8], if $c = \{c_n\} \in \gamma RBVS$ then there exists a non-increasing sequence $\gamma^* = \{\gamma_n^*\}$ such that $c \in \gamma^* RBVS$, throughout the manuscript, when using the notation $c = \{c_n\} \in \gamma RBVS$, we assume, without loss of generality, that $\gamma = \{\gamma_n\}$ is non-increasing sequence.

The mixed Lebesgue space $L_{\vec{p}}[\mathbb{T}^2]$ generalizes the standard Lebesgue space $L_p[\mathbb{T}^2]$ and is defined [8] for $\vec{p} = (p_1, p_2) \in [1, \infty]^2$. It comprises all measurable functions f satisfying

$$\|f\|_{\vec{p}} = \left(\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |f(z, w)|^{p_1} dz \right)^{\frac{p_2}{p_1}} dw \right)^{\frac{1}{p_2}} < \infty.$$

That is, first compute the L_{p_1} norm of the function $f(\cdot, w)$, then compute the L_{p_2} norm of the result (which depends on w). If either p_1 or p_2 is ∞ , the appropriate integral is replaced by an essential supremum. If we take $\vec{p} = (p, p) \in [1, \infty]^2$ then the mixed Lebesgue space $L_{\vec{p}}[\mathbb{T}^2]$ coincides with the classical Banach space $L_p[\mathbb{T}^2]$. It's noteworthy that the function $f(x, y) = \frac{1}{(1 + |xy|)^{1/2}} \in L_{\vec{p}}[\mathbb{T}^2]$ for $p_1 = 2$ and $p_2 = 4$ but $f \notin L_2[\mathbb{T}^2]$.

The space $H_{\alpha, \beta}^{\vec{p}}[\mathbb{T}^2]$ is defined as,

$$H_{\alpha, \beta}^{\vec{p}}[\mathbb{T}^2] = \left\{ f \in L_{\vec{p}}[\mathbb{T}^2] : \|f(z + s, w + t) - f(z, w)\|_{\vec{p}} = O(|t|^\alpha + |s|^\beta) \right\}$$

Note that, $(H_{\alpha, \beta}^{\vec{p}}[\mathbb{T}^2], \|\cdot\|_{\alpha, \beta}^{\vec{p}})$ is a Banach space, under the norm

$$\|f\|_{\alpha, \beta}^{\vec{p}} = \|f\|_{\vec{p}} + \sup_{s \neq 0, t \neq 0} \frac{\|f(z + s, w + t) - f(z, w)\|_{\vec{p}}}{|s|^\alpha + |t|^\beta}.$$

It is evident that the Banach space $H_{\alpha, \beta}^{\vec{p}}[\mathbb{T}^2]$ serves as a generalization of the space $H_{\alpha, \beta}$.

In our current study, we broaden the findings of X. Z. Krasniqi [4, Theorem 3.1] not only for the general class $H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$ but also for a more expansive range of sequences $\{a_{m,i}\} \in \gamma RBVS$ and $\{b_{n,j}\} \in \gamma RBVS$ with respect to $0 \leq i \leq m$ and $0 \leq j \leq n$.

2. Notations and lemmas. In the sequel, following notations are used:

$$\phi_{z,w}(s,t) = \frac{f(z+s, w+t) + f(z-s, w+t) + f(z+s, w-t)}{4} + \frac{f(z-s, w-t) - 4f(z, w)}{4},$$

$$G(u, v) = \phi_{z+s, w+t}(u, v) - \phi_{z,w}(u, v),$$

$$(2.1) \quad K_{C,n}^{\lambda}(t) = \frac{1}{\pi} \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \frac{\sin\left(k + \frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)},$$

where $C = (\tilde{c}_{n,k})$ is a lower triangular regular matrix.

Lemma 2.1 ([4]). *Let $(\tilde{c}_{n,k})$ be a lower triangular regular matrix with non negative entries. Then for $0 < t \leq \frac{1}{\lambda(n) + 1}$,*

$$K_{C,n}^{\lambda}(t) = O(\lambda(n) + 1).$$

Lemma 2.2. *Let $(\tilde{c}_{n,k})$ be a lower triangular regular matrix with non negative entries and $\{\tilde{c}_{n,k}\} \in \gamma RBVS$ with respect to k . Then for $\frac{1}{\lambda(n) + 1} < t \leq \pi$,*

$$K_{C,n}^{\lambda}(t) = O\left(\frac{A_{\lambda(n),\tau}}{t}\right),$$

where $A_{\lambda(n),\tau} = K(c, \gamma) \sum_{s=0}^{\tau} \gamma_{\lambda(n),s}$ and τ denotes the integer part of $\frac{1}{t}$.

Proof. Using Jordan's inequality $\sin(\alpha) \geq \frac{2\alpha}{\pi}$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$, we have,

$$\begin{aligned}
 |K_{C,n}^\lambda(t)| &= \left| \frac{1}{\pi} \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \frac{\sin(k + \frac{1}{2})t}{\sin(\frac{t}{2})} \right| \\
 &\leq \frac{1}{2t} \left| \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \Im \left(\exp(i(k + \frac{1}{2})t) \right) \right| \\
 (2.2) \quad &\leq \frac{1}{2t} \left| \exp\left(\frac{it}{2}\right) \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| \\
 &\leq \frac{1}{2t} \left| \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right|,
 \end{aligned}$$

where $\Im$ means the imaginary part. Further, for $\frac{1}{\lambda(n)+1} < t \leq \pi$, we write,

$$\begin{aligned}
 (2.3) \quad \left| \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| &\leq \left| \sum_{k=0}^{\tau-1} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| + \left| \sum_{k=\tau}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| \\
 &= \mathcal{S}_1 + \mathcal{S}_2
 \end{aligned}$$

Since, $\{\tilde{c}_{\lambda(n),k}\} \in \gamma RBVS$ with respect to k , we get

$$(2.4) \quad \tilde{c}_{\lambda(n),\tau} \leq \sum_{k=\tau}^{\infty} |\tilde{c}_{\lambda(n),k} - \tilde{c}_{\lambda(n),k+1}| \leq K(c, \gamma) \gamma_{\lambda(n),\tau}.$$

Using (2.4),

$$(2.5) \quad \mathcal{S}_1 = \left| \sum_{k=0}^{\tau-1} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| \leq \sum_{k=0}^{\tau-1} |\tilde{c}_{\lambda(n),k}| \leq \sum_{k=0}^{\tau-1} K(c, \gamma) \gamma_{\lambda(n),k} = A_{\lambda(n),\tau-1}.$$

Using Abel's transformation and assumption $\{\tilde{c}_{\lambda(n),k}\} \in \gamma RBVS$ with respect to k , and notation, $\mathcal{E}_k = \sum_{p=0}^k \exp(ip t)$, we get

(2.6)

$$\begin{aligned}
 \mathcal{S}_2 &= \left| \sum_{k=\tau}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| \\
 &= \left| \sum_{k=\tau}^{\lambda(n)-1} (\tilde{c}_{\lambda(n),k} - \tilde{c}_{\lambda(n),k+1}) \mathcal{E}_k - \tilde{c}_{\lambda(n),\tau} \mathcal{E}_{\tau-1} + \tilde{c}_{\lambda(n),\lambda(n)} \mathcal{E}_{\lambda(n)} \right| \\
 &\leq \sum_{k=\tau}^{\lambda(n)-1} |\tilde{c}_{\lambda(n),k} - \tilde{c}_{\lambda(n),k+1}| |\mathcal{E}_k| + |\tilde{c}_{\lambda(n),\tau}| |\mathcal{E}_{\tau-1}| + |\tilde{c}_{\lambda(n),\lambda(n)}| |\mathcal{E}_{\lambda(n)}| \\
 &\leq \left(\sum_{k=\tau}^{\lambda(n)-1} |\tilde{c}_{\lambda(n),k} - \tilde{c}_{\lambda(n),k+1}| + |\tilde{c}_{\lambda(n),\tau}| + |\tilde{c}_{\lambda(n),\lambda(n)}| \right) \times \max_{\tau-1 \leq k \leq \lambda(n)} |\mathcal{E}_k| \\
 &\leq 3K(c, \gamma) \gamma_{\lambda(n),\tau} \times \max_{\tau-1 \leq k \leq \lambda(n)} \left| \frac{1 - \exp i(k+1)t}{1 - \exp(it)} \right| \\
 &\leq \frac{3K(c, \gamma) \gamma_{\lambda(n),\tau}}{\sqrt{(1 - \cos(t))^2 + \sin^2(t)}} = \frac{3K(c, \gamma) \gamma_{\lambda(n),\tau}}{2 \sin \frac{t}{2}} = \frac{3\pi K(c, \gamma) \gamma_{\lambda(n),\tau}}{2t} \\
 &\leq \frac{3\pi}{2} K(c, \gamma) (\tau + 1) \gamma_{\lambda(n),\tau}.
 \end{aligned}$$

Now, from (2.2) to (2.6), we get

$$\begin{aligned}
 |K_{C,n}^{\lambda}(t)| &\leq \frac{1}{2t} \left| \sum_{k=0}^{\lambda(n)} \tilde{c}_{\lambda(n),k} \exp(ikt) \right| \\
 &\leq \frac{1}{2t} (A_{\lambda(n),\tau-1} + \frac{3\pi}{2} K(c, \gamma) (\tau + 1) \gamma_{\lambda(n),\tau}) = O\left(\frac{A_{\lambda(n),\tau}}{t}\right).
 \end{aligned}$$

□

The following lemma is a consequence of Minkowski's integral inequality [2, Theorem 202]:

Lemma 2.3 ([6]). *If $F(x, u)$ is a Lebesgue measurable function of two variables defined for $0 \leq u \leq \pi; 0 \leq x \leq 2\pi$, then*

$$\left\| \int_0^\pi F(x, u) du \right\|_p \leq \int_0^\pi \|F(x, u)\|_p du, \quad p \geq 1.$$

Lemma 2.4. *If $F(x, y, u, v)$ is a Lebesgue measurable function of four variables defined for $0 \leq u, v \leq \pi; 0 \leq x, y \leq 2\pi$, then*

$$\left\| \int_0^\pi \int_0^\pi F(x, y, u, v) dudv \right\|_{\vec{p}} \leq \int_0^\pi \int_0^\pi \|F(x, y, u, v)\|_{\vec{p}} dudv, \quad p_1 \geq 1 \text{ and } p_2 \geq 1.$$

Proof. When $p_1, p_2 < \infty$, apply Lemma 2.3 repeatedly in the following manner

$$\begin{aligned} & \left\| \int_0^\pi \int_0^\pi F(x, y, u, v) dudv \right\|_{\vec{p}} \\ &= \left(\int_0^{2\pi} \left(\int_0^{2\pi} \left(\left| \int_0^\pi \int_0^\pi F(x, y, u, v) dudv \right| \right)^{p_1} dx \right)^{\frac{p_2}{p_1}} dy \right)^{\frac{1}{p_2}} \\ &= \left(\int_0^{2\pi} \left(\int_0^{2\pi} \left(\left| \int_0^\pi \left(\int_0^\pi F(x, y, u, v) du \right) dv \right| \right)^{p_1} dx \right)^{\frac{1}{p_1} \times p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq \left(\int_0^{2\pi} \left(\int_0^\pi \left(\int_0^{2\pi} \left(\left| \int_0^\pi F(x, y, u, v) du \right| \right)^{p_1} dx \right)^{\frac{1}{p_1}} dv \right)^{p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq \left(\int_0^{2\pi} \left(\int_0^\pi \left(\int_0^\pi \left(\int_0^{2\pi} |F(x, y, u, v)|^{p_1} dx \right)^{\frac{1}{p_1}} du \right) dv \right)^{p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq \left(\int_0^\pi \left(\int_0^{2\pi} \left(\int_0^\pi \left(\int_0^{2\pi} |F(x, y, u, v)|^{p_1} dx \right)^{\frac{1}{p_1}} du \right)^{p_2} dy \right)^{\frac{1}{p_2}} dv \right) \\ &\leq \left(\int_0^\pi \left(\int_0^\pi \left(\int_0^{2\pi} \left(\int_0^{2\pi} |F(x, y, u, v)|^{p_1} dx \right)^{\frac{p_2}{p_1}} dy \right)^{\frac{1}{p_2}} du \right) dv \right) \end{aligned}$$

$$= \int_0^\pi \int_0^\pi \|F(x, y, u, v)\|_{\vec{p}} du dv.$$

When $p_1 = \infty$ and $p_2 < \infty$, by Lemma 2.3,

$$\begin{aligned} & \left\| \int_0^\pi \int_0^\pi F(x, y, u, v) du dv \right\|_{\vec{p}} \\ &= \left(\int_0^{2\pi} \left(\operatorname{ess\,sup}_x \left(\left| \int_0^\pi \int_0^\pi F(x, y, u, v) du dv \right| \right)^{p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq \left(\int_0^{2\pi} \left(\int_0^\pi \int_0^\pi \operatorname{ess\,sup}_x |F(x, y, u, v)| du dv \right)^{p_2} dy \right)^{\frac{1}{p_2}} \\ &\leq \int_0^\pi \int_0^\pi \left(\int_0^{2\pi} \left(\operatorname{ess\,sup}_x |F(x, y, u, v)| \right)^{p_2} dy \right)^{\frac{1}{p_2}} du dv \\ &= \int_0^\pi \int_0^\pi \|F(x, y, u, v)\|_{\vec{p}} du dv. \end{aligned}$$

When $p_1 < \infty$ and $p_2 = \infty$, by Lemma 2.3,

$$\begin{aligned} & \left\| \int_0^\pi \int_0^\pi F(x, y, u, v) du dv \right\|_{\vec{p}} \\ &= \operatorname{ess\,sup}_y \left(\int_0^{2\pi} \left(\int_0^\pi \int_0^\pi |F(x, y, u, v)| du dv \right)^{p_1} dx \right)^{\frac{1}{p_1}} \\ &\leq \operatorname{ess\,sup}_y \int_0^\pi \int_0^\pi \left(\int_0^{2\pi} |F(x, y, u, v)|^{p_1} dx \right)^{\frac{1}{p_1}} du dv \\ &\leq \int_0^\pi \int_0^\pi \operatorname{ess\,sup}_y \left(\int_0^{2\pi} |F(x, y, u, v)|^{p_1} dx \right)^{\frac{1}{p_1}} du dv \\ &= \int_0^\pi \int_0^\pi \|F(x, y, u, v)\|_{\vec{p}} du dv. \end{aligned}$$

Finally, in the case $p_1 = p_2 = \infty$, both sides are equal. $\square$

3. Main result.

Theorem 3.1. *Let $A := (a_{m,i})$ and $B := (b_{n,j})$ be lower triangular regular matrices, where $\{a_{m,i}\} \in \gamma RBVS$ and $\{b_{n,j}\} \in \gamma RBVS$ with respect to $i \leq m$ and $j \leq n$, with the condition $A_{\lambda(n),\tau} = O\left(\frac{\tau}{\lambda(n)+1}\right)$, for $\tau = \left[\frac{1}{t}\right]$, $\left(\frac{1}{\lambda(n)+1} < t \leq \pi\right)$. If $f(z, w)$ is a 2π periodic function in z and w and belonging to the class $H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$ for $0 < \alpha, \beta \leq 1$, then*

$$(3.1) \quad \|S_{m,n}^\lambda(f; z, w) - f(z, w)\|_{\alpha,\beta}^{\vec{p}} = O \begin{cases} \frac{1}{(\lambda(m)+1)^\alpha} + \frac{1}{(\lambda(n)+1)^\beta}, & \text{if } 0 < \alpha < 1, 0 < \beta < 1; \\ \frac{1}{(\lambda(m)+1)^\alpha} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } 0 < \alpha < 1, \beta = 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{1}{(\lambda(n)+1)^\beta}, & \text{if } \alpha = 1, 0 < \beta < 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } \alpha = \beta = 1. \end{cases}$$

Proof. Recall that, for the partial sum of double Fourier series $S_{m,n}(F; z, w)$, by some elementary calculations, we get

$$(3.2) \quad S_{m,n}(f; z, w) - f(z, w) = \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \phi_{z,w}(s, t) \frac{\sin(m + \frac{1}{2})s}{\sin(\frac{s}{2})} \frac{\sin(n + \frac{1}{2})t}{\sin(\frac{t}{2})} ds dt.$$

Taking double matrix means of $S_{m,n}(f; z, w) - f(z, w)$, we get

$$\begin{aligned} S_{m,n}^\lambda(f; z, w) - f(z, w) &= \sum_{p=0}^{\lambda(m)} \sum_{q=0}^{\lambda(n)} a_{\lambda(m),p} b_{\lambda(n),q} [S_{p,q}(f; z, w) - f(z, w)] \\ &= \frac{1}{\pi^2} \int_0^\pi \int_0^\pi \phi_{z,w}(s, t) \sum_{p=0}^{\lambda(m)} \sum_{q=0}^{\lambda(n)} a_{\lambda(m),p} b_{\lambda(n),q} \frac{\sin(p + \frac{1}{2})s}{\sin(\frac{s}{2})} \frac{\sin(q + \frac{1}{2})t}{\sin(\frac{t}{2})} ds dt \\ &= \int_0^\pi \int_0^\pi \phi_{z,w}(s, t) K_{A,m}^\lambda(s) K_{B,n}^\lambda(t) ds dt \end{aligned}$$

$$= I_{m,n}^{\lambda}(z, w).$$

Firstly, we estimate the $I_{m,n}^{\lambda}(z + s, w + t) - I_{m,n}^{\lambda}(z, w)$,

$$\begin{aligned} & I_{m,n}^{\lambda}(z + s, w + t) - I_{m,n}^{\lambda}(z, w) \\ &= \int_0^{\pi} \int_0^{\pi} G(u, v) K_{A,m}^{\lambda}(u) K_{B,n}^{\lambda}(v) dudv \\ &= \left(\int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} + \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} + \int_{\frac{1}{\lambda(n)+1}}^{\pi} \int_0^{\frac{1}{\lambda(m)+1}} + \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \right) \\ & \quad \times G(u, v) K_{A,m}^{\lambda}(u) K_{B,n}^{\lambda}(v) dudv \\ &= I_{m,n}^1 + I_{m,n}^2 + I_{m,n}^3 + I_{m,n}^4. \end{aligned}$$

Using Lemma 2.4, we have $\|I_{m,n}^{\lambda}(z + s, w + t) - I_{m,n}^{\lambda}(z, w)\|_{\vec{p}} \leq \|I_{m,n}^1\|_{\vec{p}} + \|I_{m,n}^2\|_{\vec{p}} + \|I_{m,n}^3\|_{\vec{p}} + \|I_{m,n}^4\|_{\vec{p}}$. Where,

$$\begin{aligned} \|I_{m,n}^1\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} \|G(u, v)\|_{\vec{p}} |K_{A,m}^{\lambda}(u)| |K_{B,n}^{\lambda}(v)| dudv, \\ \|I_{m,n}^2\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \|G(u, v)\|_{\vec{p}} |K_{A,m}^{\lambda}(u)| |K_{B,n}^{\lambda}(v)| dudv, \\ \|I_{m,n}^3\|_{\vec{p}} &= \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_0^{\frac{1}{\lambda(n)+1}} \|G(u, v)\|_{\vec{p}} |K_{A,m}^{\lambda}(u)| |K_{B,n}^{\lambda}(v)| dudv, \\ \|I_{m,n}^4\|_{\vec{p}} &= \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \|G(u, v)\|_{\vec{p}} |K_{A,m}^{\lambda}(u)| |K_{B,n}^{\lambda}(v)| dudv. \end{aligned}$$

It is easy to verify, if $f \in H_{\alpha, \beta}^{\vec{p}}[\mathbb{T}^2]$ then $\|G(u, v)\|_{\vec{p}} = O(|s|^\alpha + |t|^\beta)$.
By Lemma 2.1, we have

$$\begin{aligned}
 \|I_{m,n}^1\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} \|G(u, v)\|_{\vec{p}} |K_{A,m}^\lambda(u)| |K_{B,n}^\lambda(v)| du dv \\
 (3.3) \quad &= O(\lambda(m) + 1) O(\lambda(n) + 1) \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} \|G(u, v)\|_{\vec{p}} du dv \\
 &= O(|s|^\alpha + |t|^\beta).
 \end{aligned}$$

Using an assumption that $A_{\lambda(n), \tau} = O\left(\frac{\tau}{\lambda(n) + 1}\right)$ for $\tau = \left[\frac{1}{t}\right]$,
 $\left(\frac{1}{\lambda(n) + 1} < t \leq \pi\right)$ and with the help of Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned}
 \|I_{m,n}^2\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \|G(u, v)\|_{\vec{p}} |K_{A,m}^\lambda(u)| |K_{B,n}^\lambda(v)| du dv \\
 &= O(\lambda(m) + 1) \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \|G(u, v)\|_{\vec{p}} \frac{A_{\lambda(n), [\frac{1}{v}]}}{v} du dv \\
 (3.4) \quad &= O(|s|^\alpha + |t|^\beta) \int_{\frac{1}{\lambda(n)+1}}^{\pi} \frac{A_{\lambda(n), [\frac{1}{v}]}}{v} dv \\
 &= O(|s|^\alpha + |t|^\beta) \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{A_{\lambda(n), [t]}}{t} dt \\
 &= O(|s|^\alpha + |t|^\beta) \frac{1}{\lambda(n) + 1} \int_{\frac{1}{\pi}}^{\lambda(n)+1} dt \\
 &= O(|s|^\alpha + |t|^\beta).
 \end{aligned}$$

Similarly,

$$(3.5) \quad \|I_{m,n}^3\|_{\vec{p}} = O(|s|^\alpha + |t|^\beta),$$

as well as,

$$(3.6) \quad \|I_{m,n}^4\|_{\vec{p}} = O(|s|^\alpha + |t|^\beta).$$

Now, from equations (3.3) to (3.6), we have,

$$(3.7) \quad \sup_{s \neq 0, t \neq 0} \frac{\|I_{m,n}^\lambda(z+s, w+t) - I_{m,n}^\lambda(z, w)\|_{\vec{p}}}{|s|^\alpha + |t|^\beta} = O(1)$$

Next we estimate the remaining part,

$$(3.8) \quad \begin{aligned} I_{m,n}^\lambda(z, w) &= \int_0^\pi \int_0^\pi \phi_{z,w}(s, t) K_{A,m}^\lambda(s) K_{B,n}^\lambda(t) ds dt \\ &= \left(\int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} + \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^\pi + \int_{\frac{1}{\lambda(m)+1}}^\pi \int_0^{\frac{1}{\lambda(n)+1}} \right. \\ &\quad \left. + \int_{\frac{1}{\lambda(m)+1}}^\pi \int_{\frac{1}{\lambda(n)+1}}^\pi \right) \times \phi_{z,w}(s, t) K_{A,m}^\lambda(s) K_{B,n}^\lambda(t) ds dt \\ &= L_{m,n}^1 + L_{m,n}^2 + L_{m,n}^3 + L_{m,n}^4. \end{aligned}$$

By Lemma 2.4, we have, $\|I_{m,n}^\lambda(z, w)\|_{\vec{p}} \leq \|L_{m,n}^1\|_{\vec{p}} + \|L_{m,n}^2\|_{\vec{p}} + \|L_{m,n}^3\|_{\vec{p}} + \|L_{m,n}^4\|_{\vec{p}}$, where

$$\begin{aligned} \|L_{m,n}^1\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} \|\phi_{z,w}(s, t)\|_{\vec{p}} |K_{A,m}^\lambda(s)| |K_{B,n}^\lambda(t)| ds dt, \\ \|L_{m,n}^2\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^\pi \|\phi_{z,w}(s, t)\|_{\vec{p}} |K_{A,m}^\lambda(s)| |K_{B,n}^\lambda(t)| ds dt, \\ \|L_{m,n}^3\|_{\vec{p}} &= \int_{\frac{1}{\lambda(m)+1}}^\pi \int_0^{\frac{1}{\lambda(n)+1}} \|\phi_{z,w}(s, t)\|_{\vec{p}} |K_{A,m}^\lambda(s)| |K_{B,n}^\lambda(t)| ds dt, \\ \|L_{m,n}^4\|_{\vec{p}} &= \int_{\frac{1}{\lambda(m)+1}}^\pi \int_{\frac{1}{\lambda(n)+1}}^\pi \|\phi_{z,w}(s, t)\|_{\vec{p}} |K_{A,m}^\lambda(s)| |K_{B,n}^\lambda(t)| ds dt. \end{aligned}$$

It is easy to verify that $f \in H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$ implies $\phi_{z,w} \in H_{\alpha,\beta}^{\vec{p}}[\mathbb{T}^2]$.

From Lemma 2.1, we have,

$$(3.9) \quad \begin{aligned} \|L_{m,n}^1\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} \|\phi_{z,w}(s, t)\|_{\vec{p}} |K_{A,m}^\lambda(s)| |K_{B,n}^\lambda(t)| ds dt \\ &= O(\lambda(m) + 1) O(\lambda(n) + 1) \int_0^{\frac{1}{\lambda(m)+1}} \int_0^{\frac{1}{\lambda(n)+1}} (s^\alpha + t^\beta) ds dt \\ &= O\left((\lambda(m) + 1)^{-\alpha} + (\lambda(n) + 1)^{-\beta}\right) \end{aligned}$$

Using Lemma 2.1 and Lemma 2.2 and the assumption $A_{\lambda(n),\tau} = O\left(\frac{\tau}{\lambda(n)+1}\right)$ for $\tau = \left[\frac{1}{t}\right]$, $\left(\frac{1}{\lambda(n)+1} < t \leq \pi\right)$, we get

(3.10)

$$\begin{aligned}
\|L_{m,n}^2\|_{\vec{p}} &= \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} \|\phi_{z,w}(s,t)\|_{\vec{p}} |K_{A,m}^{\lambda}(s)| |K_{B,n}^{\lambda}(t)| ds dt \\
&= O\left((\lambda(m)+1) \int_0^{\frac{1}{\lambda(m)+1}} \int_{\frac{1}{\lambda(n)+1}}^{\pi} (s^{\alpha} + t^{\beta}) \frac{A_{\lambda(n),[\frac{1}{t}]} }{t} ds dt\right) \\
&= O\left((\lambda(m)+1) \left[\int_0^{\frac{1}{\lambda(m)+1}} s^{\alpha} ds \int_{\frac{1}{\lambda(n)+1}}^{\pi} \frac{A_{\lambda(n),[\frac{1}{t}]} }{t} dt \right. \right. \\
&\quad \left. \left. + \int_0^{\frac{1}{\lambda(m)+1}} ds \int_{\frac{1}{\lambda(n)+1}}^{\pi} t^{\beta-1} A_{\lambda(n),[\frac{1}{t}]} dt \right] \right) \\
&= O\left((\lambda(m)+1) \left[(\lambda(m)+1)^{-\alpha-1} \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{A_{\lambda(n),[v]} }{v} dv \right. \right. \\
&\quad \left. \left. + \frac{1}{\lambda(m)+1} \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{A_{\lambda(n),[v]} }{v^{\beta+1}} dv \right] \right) \\
&= O\left((\lambda(m)+1) \left[(\lambda(m)+1)^{-\alpha-1} \right. \right. \\
&\quad \left. \left. + \frac{1}{\lambda(m)+1} \frac{1}{\lambda(n)+1} \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{1}{v^{\beta}} dv \right] \right) \\
&= O\left\{ \begin{array}{ll} \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if } 0 < \alpha \leq 1, 0 < \beta < 1; \\ \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } 0 < \alpha \leq 1, \beta = 1. \end{array} \right.
\end{aligned}$$

Equipped with Lemma 2.1 and 2.2, as well as the assumption, we get

(3.11)

$$\begin{aligned}
 \|L_{m,n}^3\|_{\vec{p}} &= \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_0^{\frac{1}{\lambda(n)+1}} \|\phi_{z,w}(s,t)\|_{\vec{p}} |K_{A,m}^{\lambda}(s)| |K_{B,n}^{\lambda}(t)| ds dt \\
 &= O \left((\lambda(n) + 1) \int_{\frac{1}{\lambda(m)+1}}^{\pi} \int_0^{\frac{1}{\lambda(n)+1}} (s^{\alpha} + t^{\beta}) \frac{A_{\lambda(m),[\frac{1}{s}]} }{s} ds dt \right) \\
 &= O \left((\lambda(n) + 1) \left[\frac{1}{\lambda(n) + 1} \int_{\frac{1}{\lambda(m)+1}}^{\pi} s^{\alpha-1} A_{\lambda(m),[\frac{1}{s}]} ds \right. \right. \\
 &\quad \left. \left. + \int_{\frac{1}{\lambda(m)+1}}^{\pi} \frac{A_{\lambda(m),[\frac{1}{s}]} }{s} ds \int_0^{\frac{1}{\lambda(n)+1}} t^{\beta} dt \right] \right) \\
 &= O \left((\lambda(n) + 1) \left[\frac{1}{\lambda(n) + 1} \int_{\frac{1}{\pi}}^{\lambda(m)+1} \frac{A_{\lambda(m),[u]} }{u^{\alpha+1}} du \right. \right. \\
 &\quad \left. \left. + [\lambda(n) + 1]^{-\beta-1} \int_{\frac{1}{\pi}}^{\lambda(m)+1} \frac{A_{\lambda(m),[u]} }{u} dv \right] \right) \\
 &= O \left((\lambda(n) + 1) \left[\frac{1}{\lambda(n) + 1} \frac{1}{\lambda(m) + 1} \int_{\frac{1}{\pi}}^{\lambda(m)+1} \frac{du}{u^{\alpha}} \right. \right. \\
 &\quad \left. \left. + [\lambda(n) + 1]^{-\beta-1} \right] \right) \\
 &= O \begin{cases} \frac{1}{(\lambda(m) + 1)^{\alpha}} + \frac{1}{(\lambda(n) + 1)^{\beta}}, & \text{if } 0 < \alpha < 1, 0 < \beta \leq 1; \\ \frac{\log(\lambda(m) + 1)}{\lambda(m) + 1} + \frac{1}{(\lambda(n) + 1)^{\beta}}, & \text{if } \alpha = 1, 0 < \beta \leq 1. \end{cases}
 \end{aligned}$$

Now from assumption and Lemma 2.2, we have

(3.12)

$$\begin{aligned}
\|L_{m,n}^4\|_{\vec{p}} &= \int_{\lambda(m)+1}^{\pi} \int_{\lambda(n)+1}^{\pi} \|\phi_{z,w}(s,t)\|_{\vec{p}} |K_{A,m}^{\lambda}(s)| |K_{B,n}^{\lambda}(t)| ds dt \\
&= O \left(\int_{\lambda(m)+1}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} (s^{\alpha} + t^{\beta}) \frac{A_{\lambda(m),[\frac{1}{s}]}^{\lambda}}{s} \frac{A_{\lambda(n),[\frac{1}{t}]}^{\lambda}}{t} ds dt \right) \\
&= O \left(\int_{\lambda(m)+1}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} s^{\alpha-1} A_{\lambda(m),[\frac{1}{s}]}^{\lambda} \frac{A_{\lambda(n),[\frac{1}{t}]}^{\lambda}}{t} ds dt \right. \\
&\quad \left. + \int_{\lambda(m)+1}^{\pi} \int_{\frac{1}{\lambda(n)+1}}^{\pi} t^{\beta-1} A_{\lambda(n),[\frac{1}{t}]}^{\lambda} \frac{A_{\lambda(m),[\frac{1}{s}]}^{\lambda}}{s} ds dt \right) \\
&= O \left(\int_{\frac{1}{\pi}}^{\lambda(m)+1} \frac{A_{\lambda(m),[u]}^{\lambda}}{u^{\alpha+1}} du \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{A_{\lambda(n),[v]}^{\lambda}}{v} dv \right. \\
&\quad \left. + \int_{\frac{1}{\pi}}^{\lambda(m)+1} \frac{A_{\lambda(m),[u]}^{\lambda}}{u} du \int_{\frac{1}{\pi}}^{\lambda(n)+1} \frac{A_{\lambda(n),[v]}^{\lambda}}{v^{\beta+1}} dv \right) \\
&= O \begin{cases} \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if } 0 < \alpha < 1, 0 < \beta < 1; \\ \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } 0 < \alpha < 1, \beta = 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if } \alpha = 1, 0 < \beta < 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } \alpha = \beta = 1. \end{cases}
\end{aligned}$$

From equation (3.9) to (3.12), we get

(3.13)

$$\|I_{m,n}^{\lambda}(z, w)\|_{\vec{p}} = O \begin{cases} \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if } 0 < \alpha < 1, 0 < \beta < 1; \\ \frac{1}{(\lambda(m)+1)^{\alpha}} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } 0 < \alpha < 1, \beta = 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{1}{(\lambda(n)+1)^{\beta}}, & \text{if } \alpha = 1, 0 < \beta < 1; \\ \frac{\log(\lambda(m)+1)}{\lambda(m)+1} + \frac{\log(\lambda(n)+1)}{\lambda(n)+1}, & \text{if } \alpha = \beta = 1. \end{cases}$$

Using equation (3.7) and (3.13), we get (3.1). This completes the proof. $\square$

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