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A CLASS OF SOLUTIONS OF THE n -DIMENSIONAL GENERALIZED HELMHOLTZ EQUATION WHICH DESCRIBES GENERALIZED WEINGARTEN HYPERSURFACES

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ABSTRACT. In this paper, we introduce the n -dimensional generalized Helmholtz equation and present explicit solutions to this equation in terms of biharmonic functions, in particular, we get solutions that depend on holomorphic functions. Also, we present explicit radial solutions for this equation and we provide explicit solutions to the n -dimensional Helmholtz equation. In addition, as an application we introduced two classes of generalized Weingarten hypersurfaces, namely, the RSHGW-hypersurfaces and the RSGW-hypersurfaces, associated with solutions of the n -dimensional generalized Helmholtz equation and classify the RSHGW-hypersurfaces of rotation. For $n = 2$, we obtain a Weierstrass type representation for these surfaces which depend of three holomorphic functions and we classify the RSHGW-surfaces and the RSGW-surfaces of rotation.

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1. Introduction. The reduced Helmholtz equation is an elliptic differential equation allowing to describe physical phenomena related to oscillatory problems. As such, it is a classical problem for diverse fields of physics and engineering such as vibration mechanics, electro-magnetics, acoustics or quantum mechanics (see [3, 13, 18]). Solution of the two-dimensional Helmholtz equation allows to identify vibration modes for a two-dimensional domain. Analytical solutions are limited to domains with a particular shape such as a rectangle or circle (see [13, 16]). In general, solving this differential equation relies on numerical methods (see [27, 28]). In order to find solutions, the Helmholtz equation is transformed to account for a conformal map between the shape of the physical domain and the unit disk as canonical domain. This way, the transformed Helmholtz equation is solved exploiting well known analytical solutions for a circular domain and the solution in the physical domain is obtained by applying the conformal map (see [12]). Recently, the Helmholtz equation was studied by the method of fundamental solutions (MFS) using Bessel and Neumann functions. The bounds of errors are derived for bounded simply-connected domains, while the bounds of condition number are derived only for disk domains, see [14]. Also, by using conformal mappings of a plane with elliptic hole and a plane with cross-shaped hole into the exterior of the unit disk, are constructed functions that play the role of bases in the spaces of functions analytic in these domains. In addition, on the basis of expansions of analytic functions in series in these bases, are obtained solutions of the Helmholtz equation in a plane with holes whose boundary values coincide with the boundary values of these functions (see [24]).

In [16], is proposed the Galerkin boundary element method for exterior problems of the two dimensional Helmholtz equation with arbitrary wavenumber. In 2002, is presented a new boundary integral method for solving the general Helmholtz equation is developed. This new formulation is developed for the two dimensional Helmholtz equation with the method of moments Laplacian solution. The main feature of this new formulation is that the boundary conditions are satisfied independent of the region node discretizations, see [23].

In 1934, Eisenhart [8] showed that in the Euclidean 3-space, the Helmholtz equation separates only in eleven coordinate systems, which sometimes are called the Eisenhart coordinate systems:

1. Rectangular coordinates, 2. Circular-cylinder coordinates, 3. Elliptic-cylinder coordinates, 4. Parabolic-cylinder coordinates, 5. Spherical coordinates 6. Prolate spheroidal coordinates, 7. Oblate spheroidal coordinates, 8. Parabolic coordinates, 9. Conical coordinates, 10. Ellipsoidal coordinates, 11. Paraboloidal coordinates.

If the Helmholtz equation separates, we can solve the ordinary

differential equations individually and combine the results to find the exact solution of the Helmholtz equation. So for a given problem involving the Helmholtz equation, we can find a suitable coordinate system which allows separability and therefore we can find the exact solution to the problem.

The two-dimensional generalized Helmholtz equation is introduced and studied in [20], explicit solutions to this equation were presented. As an application, explicit solutions to the two-dimensional Helmholtz equation are obtained.

In this paper, given the function $h : \mathbb{R}^n \rightarrow \mathbb{R}$, $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ we consider the following equations:

The n -dimensional Helmholtz equation for h is defined as

$$(1.1) \quad \Delta h(u) + c(\Omega(u))^2 h(u) = 0,$$

The n -dimensional generalized Helmholtz equation for h is defined as

$$(1.2) \quad \Delta \left(\frac{1}{(\Omega(u))^2} (\Delta h(u) + c(\Omega(u))^2 h(u)) \right) = 0,$$

where $\Omega(u)$ is a differentiable function and c is a real non-zero constant.

In general, given a partial differential equation (PDE) it is not easy to find explicit solutions for this PDE, here for $\Omega(u) = \frac{2\sqrt{2}}{a + b\|u\|^2}$ and $c = ab$ we obtain a class of solutions for the equation (1.2) that depend on biharmonic functions and a class of solutions for the equation (1.1). Note that every solution of the n -dimensional Helmholtz equation is a solution of the n -dimensional generalized Helmholtz equation.

Let $M \subset \mathbb{R}^3$ be a surface oriented by its normal Gauss map N . Given $\nu \in \mathbb{R}^3$, the functions $\Psi_\nu, \Lambda_\nu : M \rightarrow \mathbb{R}$ given by $\Psi_\nu(p) = \langle p - \nu, N(p) \rangle$, $\Lambda_\nu(p) = \langle p - \nu, p - \nu \rangle$, $p \in M$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$, are called *support function* and *quadratic distance function* with respect to $\nu \in \mathbb{R}^3$, respectively. Geometrically, $\Psi_\nu(p)$ measures the signed distance from ν to the tangent plane $T_p M$ of M in p and $\Lambda_\nu(p)$ measures the square of the distance from p to ν .

In 1907, Tzitzéica [26] studied oriented hyperbolic surfaces such that there exist a nonzero constant $c \in \mathbb{R}$ for which $K + c^2 \Psi^4 = 0$. In 1888, Appell [1] studied a class of oriented surfaces in $\mathbb{R}^3$ associated with area preserving transformations in the sphere. After, Ferreira and Roitman [11] showed that these surfaces are such that there is a fixed point $\nu \in \mathbb{R}^3$ such that the mean curvature H , the Gaussian curvature K and the support function Ψ_ν satisfy $H + \Psi_\nu K = 0$. Dias and Corro [6], study a special class of oriented surfaces $M \subset \mathbb{R}^3$ that satisfy a relation of the form $2\Psi_\nu H + \Lambda_\nu K = 0$, for some fixed point $\nu \in \mathbb{R}^3$. Called

generalized special Weingarten surfaces depending on support function and the distance function (in short, EDSGW-surface), this class of surfaces has the geometric property that all the middle spheres pass through a fixed point. Also, is obtained a Weierstrass type representation for this surfaces that depend on two holomorphic functions. As applications classifies the EDSGW-surface of rotation and present a 2-parameter family of complete cyclic EDSGW-surfaces with an isolated singularity and foliated by non-parallel planes, which shows that one can not generalize the result of López [15] valid for linear Weingarten surfaces. Also, in [4], it is shown that the EDSGW-surfaces are in correspondence with the class of surfaces in $\mathbb{S}^2 \times \mathbb{R}$ whose Gaussian curvature K and extrinsic curvature K_E satisfy $K = 2K_E$.

Martinez and Roitman [17], introduce the class of surfaces in Euclidean space motivated by a problem posed by Élie Cartan. This class of surfaces are called *Ribaucour surfaces* and are defined as surfaces where all the middle spheres intercept a fixed sphere along a large circle, these surfaces satisfy a relation of the form $2\Psi H + (1 + \Lambda)K = 0$. The authors obtain holomorphic data for these surfaces and discuss the relation with minimal surfaces. Ribaucour surfaces are mentioned in passant by Bianchi [2].

Also, in this paper, thus as the two-dimensional Helmholtz equation has several applications for diverse fields of physics and engineering such as vibration mechanics, electro-magnetics, acoustics or quantum mechanics, we present geometrical applications of the equations (1.1) and (1.2), namely, we introduced two classes of generalized Weingarten hypersurfaces associated with solutions of the equations (1.1) and (1.2), defined as:

An orientable hypersurface $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, is called *generalized Weingarten hypersurface depending on support function and the radius function* (in short, RSGW-hypersurface) if satisfy

$$n \frac{H_{n-1}}{H_n} - nR - 2\Psi = 0.$$

An orientable hypersurface $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, is called *harmonic generalized Weingarten hypersurface depending on support function and the radius function* (in short RSHGW-hypersurface) if satisfy

$$\Delta \left(\frac{R}{d} \left(n \frac{H_{n-1}}{H_n} - nR - 2\Psi \right) \right) = 0,$$

where H_{n-1}, H_n are the r th-mean curvatures, Ψ is the support function with relation to origin, R is the radius function with respect to hyperplane $\Pi = \{(u_1, u_2, \dots, u_n, 0)/u_i \in \mathbb{R}\}$ and d is the projection function of M on Π .

The paper is organized as follows. In Section 2, we present the definitions

and results that will be used in the work. In Section 3, we present our main results, namely, we introduce the n -dimensional generalized Helmholtz equation and present explicit solutions to this equation in terms of biharmonic functions, in particular, we get solutions that depend on holomorphic functions. Also, we provide explicit radial solutions for this equation. Moreover, we present explicit solutions for the n -dimensional Helmholtz equation. In Section 4, as an application we introduced two classes of generalized Weingarten hypersurfaces, namely, the RSHGW-hypersurfaces and the RSGW-hypersurfaces, associated with solutions of the n -dimensional generalized Helmholtz equation and classify the RSHGW-hypersurfaces of rotation. In Section 5, for the case the surfaces, we obtain a Weierstrass type representation for these surfaces which depend of three holomorphic functions. Finally, we classify the RSHGW-surfaces and the RSGW-surfaces of rotation.

2. Preliminaries. In this section we present the definitions and results that will be used in the work. In [23], is considered the following two-dimensional elliptic equation for a smooth function $\Phi(u_1, u_2)$ defined in a region defined by $\mathfrak{R}$ which is bounded by a contour C so that

$$(2.1) \quad \Delta\Phi(u_1, u_2) + \lambda(u_1, u_2)\Phi(u_1, u_2) = F(u_1, u_2),$$

where $\lambda(u_1, u_2)$ and $F(u_1, u_2)$ are known functions in the domain $\mathfrak{R}$.

The general form of (2.1) includes, as specializations, the following cases:

1. Laplace's equation, with $\lambda = 0$ and $F = 0$,
2. Poisson's equation, with $\lambda = 0$ and $F \neq 0$,
3. Helmholtz's equation, with $\lambda \neq 0$ and $F \neq 0$ ($F = 0$).

The *two-dimensional Helmholtz equation* for $h(u)$ defined by (see [20])

$$(2.2) \quad \Delta h(u) + c(\Omega(u))^2 h(u) = 0,$$

where $\Omega(u)$ indicates the wavenumber and c is a real non-zero constant.

Definition 2.1. The two-dimensional generalized Helmholtz equation for $h(u)$ is defined as (see [20])

$$(2.3) \quad \Delta \left(\frac{1}{(\Omega(u))^2} (\Delta h(u) + c(\Omega(u))^2 h(u)) \right) = 0,$$

where $\Omega(u)$ is a function and c is a real non-zero constant.

The following Lemma is an equivalent version to the Lemma 1 obtained in [21]

Lemma 2.2. *If $f_1, f_2, g : \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions of $u = u_1 + iu_2$, such that $\langle 1, f_1 \rangle + \langle g, f_2 \rangle = 0$. Then $f_1 = -\overline{z_1}g + ic_1$, $f_2 = ic_2g + z_1$, where c_i are real constants and $z_1 \in \mathbb{C}$.*

Definition 2.3. Let $M \subset \mathbb{R}^{n+1}$ be an orientable hypersurface with normal Gauss map N . Given $\nu \in \mathbb{R}^{n+1}$, the function $\Psi_\nu : M \rightarrow \mathbb{R}^{n+1}$ given by $\Psi_\nu(p) = \langle p - \nu, N(p) \rangle$, $p \in M$, where $\langle, \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^{n+1}$ is called *support function* with respect to $\nu \in \mathbb{R}^{n+1}$. Geometrically, $\Psi_\nu(p)$ measures the signed distance from ν to the tangent hyperplane $T_p M$ of M in p .

Definition 2.4. Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, the *Mean curvature* and the *Gauss-Kronecker curvature* of M are given by

$$H = \frac{1}{n} \sum_{i=1}^n k_i, \quad K = \prod_{i=1}^n k_i,$$

where k_i are the principal curvatures of M .

Definition 2.5. The *rth-mean curvature* H_r of M is defined by

$$H_r = \frac{S_r(W)}{\binom{n}{r}},$$

where, for intergers $0 \leq r \leq n$, $S_r(W)$ is defined by

$$\begin{aligned} S_0(W) &= 1, \\ S_r(W) &= \sum_{1 \leq i_1 < \dots < i_r \leq n} k_{i_1} \cdots k_{i_r}, \end{aligned}$$

W is the Weingarten matrix.

We observe that $H = H_1$ and $K = H_n$.

The following results due to Ferapontov on systems of hydrodynamic type and reciprocal transformations can be seen in [10] and [9].

Definition 2.6. Let $U \subset \mathbb{R}^n$ be an open subset, $u = (u^1, \dots, u^n) \in U$ and $V_{ij} : U \rightarrow \mathbb{R}$, $1 \leq i, j \leq n$, differentiable functions. A *system of hydrodynamic type* in the independent variables (x, t) is a system of partial differential equations of the form

$$(2.4) \quad u_{,t}^i = \sum_{j=1}^n V_{ij}(u) u_{,x}^j, \quad 1 \leq i \leq n,$$

where $u^i = u^i(x, t)$, $1 \leq i \leq n$.

Definition 2.7. Let $A dx + B dt$ and $C dx + D dt$ be integrals of the system of hydrodynamic type (2.4), where $AD - BC \neq 0$ for all $u \in U$. A *reciprocal transformation* is a change in the independent variables (x, t) , given by

$$(2.5) \quad d\bar{x} = A dx + B dt$$

$$(2.6) \quad d\bar{t} = C dx + D dt$$

Proposition 2.8. *The reciprocal transformation given by (2.5) and (2.6) transform the system (2.4) in the following system of hydrodynamic type*

$$(2.7) \quad u_{,\bar{t}}^i = \sum_{j=1}^n \bar{V}_{ij}(u) u_{,\bar{x}}^j, \quad i = 1, \dots, n,$$

where the matrix $\bar{V} = (\bar{V}_{ij})$ is given by

$$(2.8) \quad \bar{V} = (AV - BI_n)(DI_n - CV)^{-1}$$

and I_n is the identity matrix $n \times n$.

Moreover, given an integral $P dx + Q dt$ of the system of hydrodynamic type (2.4), the 1-form

$$\frac{DP - CQ}{AD - BC} d\bar{x} + \frac{AQ - BP}{AD - BC} d\bar{t}$$

is an integral of hydrodynamic type of the system (2.7).

Consider the unit sphere $\mathbb{S}^n = \{q \in \mathbb{R}^{n+1}; |q| = 1\}$. We say that $e_{n+1} = (0, \dots, 0, 1)$ and $-e_{n+1} = (0, \dots, 0, -1)$ are, respectively, the north pole and south pole of $\mathbb{S}^n$. The stereographic projections $P_- : \mathbb{S}^n \setminus \{-e_{n+1}\} \rightarrow \mathbb{R}^n$ and $P_+ : \mathbb{S}^n \setminus \{e_{n+1}\} \rightarrow \mathbb{R}^n$ are diffeomorphisms given by,

$$(2.9) \quad P_-(q) = \frac{q - \langle q, e_{n+1} \rangle e_{n+1}}{1 + \langle q, e_{n+1} \rangle} \quad \text{and} \quad P_+(q) = \frac{q - \langle q, e_{n+1} \rangle e_{n+1}}{1 - \langle q, e_{n+1} \rangle}, \quad q \in \mathbb{S}^n.$$

According to Dubrovin and Novikov [7], given a differentiable function $h : U \rightarrow \mathbb{R}$, the system of hydrodynamic type associated with the zero curvature metric L_{ij} in $\mathbb{R}^n$ is given by the matrix $V = (V_{ij})$, where

$$(2.10) \quad V_{ij} = \frac{1}{L_{jj}} \left(h_{,ij} - \sum_{l=1}^n \Gamma_{ij}^l h_{,l} \right), \quad 1 \leq i, j \leq n.$$

The following results can be found in [5].

Proposition 2.9. *Let Σ be an oriented hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K(p) \neq 0$, for all $p \in \Sigma$ and the hyperplane $\Pi = \{(u_1, u_2, \dots, u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} = 0\}$. If the normal Gauss map N of Σ is such that $N(p) \neq -e_{n+1}$, for all $p \in \Sigma$, then Σ and Π are locally associated by a sphere congruence.*

Lemma 2.10. *Let $Y : U \rightarrow \Pi$ be a local orthogonal parametrization of hyperplane Π and $a_j : U \rightarrow \mathbb{R}$, $1 \leq j \leq n$, differentiable functions. Then*

$$\left(\sum_{j=1}^n \frac{a_j}{L_{jj}} Y_{,j} \right)_{,i} = \sum_{j=1}^n \frac{1}{L_{jj}} \left(a_{j,i} - \sum_{k=1}^n \Gamma_{ij}^k a_k \right) Y_{,j}, \quad i = 1, \dots, n.$$

where $L_{jj} = \langle Y_{,j}, Y_{,j} \rangle$ and Γ_{ij}^k are the Christoffel symbols of L_{ij} .

In the next Theorem we present a way to obtain hypersurfaces as envelope of a sphere congruence related to systems of hydrodynamic type, this result was obtained in [5] and generalizes the Theorem 1.2 obtained in [6].

Theorem 2.11. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$, $n \geq 2$, with Gauss-Kronecker curvature K non-zero everywhere and Gauss map $N \neq -e_{n+1}$. There exists a local orthogonal parametrization $Y : U \rightarrow \Pi$, where U is connected open subset of $\mathbb{R}^n$, $\Pi = \{(u_1, u_2, \dots, u_{n+1}) \in \mathbb{R}^{n+1} : u_{n+1} = 0\}$ and differentiable function $h : U \rightarrow \mathbb{R}$, such that M is locally parametrized by*

$$(2.11) \quad X(u) = \left(Q - \frac{2R}{T} Y, -\frac{2R}{T} \right),$$

where $u = (u_1, \dots, u_n) \in U \subset \mathbb{R}^n$,

$$(2.12) \quad T = 1 + \|Y\|^2, \quad Q = \sum_{i=1}^n \frac{h_{,i}}{L_i} Y_{,i}, \quad L_i = \langle Y_{,i}, Y_{,i} \rangle, \quad R = \langle Q, Y \rangle - h.$$

The Gauss map is given by

$$(2.13) \quad N(u) = \frac{1}{1 + \|Y\|^2} (2Y, 1 - \|Y\|^2).$$

The Weingarten matrix W is given by

$$(2.14) \quad W = 2(TV - 2RI)^{-1},$$

where I is identity matrix and the matrix $V = (V_{ij})$ is defined by

$$(2.15) \quad V_{ij} = \frac{1}{L_j} \left(h_{,ij} - \sum_{l=1}^n \Gamma_{ij}^l h_{,l} \right)$$

$$(2.16) \quad \Gamma_{ii}^i = \frac{L_{i,i}}{2L_i}, \quad \Gamma_{ij}^i = \frac{L_{i,j}}{2L_i} = -\frac{L_j}{L_i} \Gamma_{ii}^j, \quad i \neq j.$$

The regularity condition of X is given by

$$(2.17) \quad P = \det(TV - 2RI) \neq 0.$$

Moreover, the coefficients of the first, second and third fundamental form of X are given by

$$(2.18) \quad a_{ii} = \frac{L_i}{T^2} A_i^2 + \sum_{k \neq i}^n (TV_{ik})^2 L_k,$$

$$a_{ij} = -\frac{L_j}{T} V_{ij} [A_i + A_j] + \sum_{k \neq i, j}^n V_{ik} V_{jk} L_k, \quad 1 \leq i \neq j \leq n,$$

$$(2.19) \quad b_{ii} = \frac{2L_i}{T^2} A_i, \quad b_{ij} = -\frac{2}{T} L_j V_{ij}, \quad 1 \leq i \neq j \leq n, \quad A_i = 2R - TV_{ii},$$

$$(2.20) \quad \langle N_{,i}, N_{,i} \rangle = \frac{4}{T^2} L_i, \quad \langle N_{,i}, N_{,j} \rangle = 0, \quad 1 \leq i \neq j \leq n.$$

Conversely, let $Y : U \rightarrow \Pi$ be an orthogonal parametrization of Π , where U is a connected open subset of $\mathbb{R}^n$ and a differentiable function $h : U \rightarrow \mathbb{R}$. Then (2.11) define an immersion in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature non-zero, Gauss map given by (2.13) and (2.14)–(2.20) are satisfied.

Proof. Let P_- be a stereographic projection given in (2.9) and $X : U \rightarrow \Sigma$ a local parametrization of Σ . The function $Y = P_- \circ N \circ X$ define a local orthogonal parametrization of the hyperplane Π . In these coordinates, we can write $N = P_-^{-1} \circ Y$, from which it follows that N is given locally by (2.13). As $K \neq 0$, it follows from Proposition 2.9 that Σ and Π are locally associated by a sphere congruence. Thus, there are differentiable functions $a_j, R : U \rightarrow \mathbb{R}$, $1 \leq j \leq n$ such that

$$(2.21) \quad X + RN = Q + R(-e_{n+1})$$

where $Q = \sum_{j=1}^n \frac{a_j}{L_{jj}} Y_{,j}$ and $L_{jj} = \langle Y_{,j}, Y_{,j} \rangle$.

Note that $X_{,i} = Q_{,i} - R_{,i}(N + e_{n+1}) - RN_{,i}$, $1 \leq i \leq n$. Moreover, from $\langle N, X_{,i} \rangle = 0$, it follows that

$$(2.22) \quad R_{,i} = \langle Y, Q_{,i} \rangle, \quad 1 \leq i \leq n.$$

How R is integrable we have $\langle Y_{,j}, Q_{,i} \rangle = \langle Y_{,i}, Q_{,j} \rangle$ for all $1 \leq i, j \leq n$.

From Lemma 2.10,

$$(2.23) \quad Q_{,i} = \sum_{j=1}^n \frac{1}{L_{jj}} \left(a_{j,i} - \sum_{k=1}^n \Gamma_{ij}^k a_k \right) Y_{,j}, \quad 1 \leq i \leq n,$$

whence it follows that

$$(2.24) \quad \langle Y_{,j}, Q_{,i} \rangle = a_{j,i} - \sum_{k=1}^n \Gamma_{ij}^k a_k \quad \text{and} \quad \langle Y_{,i}, Q_{,j} \rangle = a_{i,j} - \sum_{k=1}^n \Gamma_{ji}^k a_k.$$

Thus, $a_{i,j} = a_{j,i}$ for all $1 \leq i, j \leq n$. Let $h : U \rightarrow \mathbb{R}$ be a possibly multivalued differentiable function such that $h_{,j} = a_j$ for all $1 \leq j \leq n$. The function h is unique up to a constant, which will be fixed later, this ensures that

$$(2.25) \quad Q = \sum_{j=1}^n \frac{h_{,j}}{L_{jj}} Y_{,j} \quad \text{and} \quad Q_{,i} = \sum_{j=1}^n V_{ij} Y_{,j}, \quad i = 1, \dots, n$$

where $V_{ij} = \frac{1}{L_{jj}} \left(h_{,ij} - \sum_{l=1}^n \Gamma_{ij}^l h_{,l} \right)$.

Now, note that $\langle Q, Y_{,i} \rangle = h_{,i}$. Using (2.22), we have

$$\langle Y, Q \rangle_{,i} = \langle Y_{,i}, Q \rangle + \langle Y, Q_{,i} \rangle = h_{,i} + R_{,i}, \quad 1 \leq i \leq n.$$

Choosing a suitable constant for h , we conclude that

$$(2.26) \quad R = \langle Q, Y \rangle - h \quad \text{and} \quad R_{,i} = \frac{1}{2} \sum_{j=1}^n V_{ij} T_{,j}, \quad i = 1, \dots, n.$$

In particular, this shows that h is not multivalued. Moreover, from (2.21) we conclude that X is given by (2.11).

From expressions (2.25) and (2.26), we obtain

$$(2.27) \quad \omega_1 = (Y + e_{n+1})dx + Qdt \quad \text{and} \quad \omega_2 = \frac{T}{2}dx + Rdt$$

are integrals of the system of hydrodynamic type (2.10), i.e.,

$$(2.28) \quad u_{,t}^i = \sum_{j=1}^n \frac{1}{L_{jj}} \left(h_{,ij} - \sum_{l=1}^n \Gamma_{ij}^l h_{,l} \right) u_{,x}^j, \quad i = 1, \dots, n.$$

Considering the reciprocal transformation given by

$$(2.29) \quad d\bar{x} = -2dt,$$

$$(2.30) \quad d\bar{t} = -2\omega_2.$$

It follows from Proposition 2.8 that (2.29) and (2.30) transform the system (2.28) in the system described by the matrix

$$(2.31) \quad \bar{V} = 2(TV - 2RI_n)^{-1}.$$

Also, the integral ω_1 is transformed into

$$(2.32) \quad \bar{\omega}_1 = \left[-\frac{1}{2}Q + \frac{R}{T}(Y + e_{n+1}) \right] d\bar{x} - \frac{(Y + e_{n+1})}{T} d\bar{t}.$$

From $-2\bar{\omega}_1 = Xd\bar{x} + (N + e_{n+1})d\bar{t}$ we obtain that

$$(2.33) \quad N_{,i} = (N + e_{n+1})_{,i} = \sum_{j=1}^n \bar{V}_{ij} X_{,j}, \quad 1 \leq i \leq n.$$

Thus, the Weingarten matrix of X is given by (2.14) and X is regular if, and only if, $P = \det(TV - 2RI_n) \neq 0$.

As $T = 1 + |Y|^2$, we can rewrite

$$(2.34) \quad N = \frac{1}{T} (2Y(u), 2 - T).$$

Thus,

$$N_{,i} = \frac{2}{T^2} (TY_{,i} - 2Y \langle Y, Y_{,i} \rangle, -2 \langle Y, Y_{,i} \rangle), \quad 1 \leq i \leq n.$$

From this expression, we obtain that the coefficients the third fundamental form is given by (2.20).

From $W = 2(TV - 2RI_n)^{-1}$, we have that

$$(2.35) \quad X_{,i} = \frac{1}{2} \sum_{k=1}^n (TV_{ik} - 2R\delta_{ik})N_{,k}, \quad 1 \leq i \leq n,$$

where the matrix $V = (V_{ij})$ is given by (2.10).

Using (2.20), it follows that the coefficients of the second fundamental form is given by

$$-\langle X_{,i}, N_{,j} \rangle = -\frac{2}{T^2} (TV_{ij} - 2R\delta_{ij})L_{jj} = \frac{2}{T^2} (2RL_{ij} - TV_{ij}L_{jj}), \quad 1 \leq i, j \leq n,$$

this equation is equivalent to (2.19).

To obtain the coefficients of the first fundamental form we use (2.20) and (2.35).

Thus,

$$\begin{aligned} \langle X_{,i}, X_{,j} \rangle &= \left\langle \frac{1}{2} \sum_{k=1}^n (TV_{ik} - 2R\delta_{ik})N_{,k}, \frac{1}{2} \sum_{l=1}^n (TV_{jl} - 2R\delta_{jl})N_{,l} \right\rangle \\ &= \sum_{k=1}^n V_{ik}V_{jk}L_{kk} - \frac{2R}{T} (V_{ij}L_{jj} + V_{ji}L_{ii}) + \frac{4R^2}{T^2} \sum_{k=1}^n \delta_{ik}\delta_{jk}L_{kk}, \\ &= \sum_{k=1}^n V_{ik}V_{jk}L_{kk} - \frac{2R}{T} (V_{ij}L_{jj} + V_{ji}L_{ii}) + \frac{4R^2}{T^2} L_{ij}, \quad 1 \leq i, j \leq n, \end{aligned}$$

this equation is equivalent to (2.18).

Conversely, note that (2.17) is a necessary and sufficient condition for (2.11) to be an immersion. In this case, it follows from (2.33) that its Gauss map is given by (2.13), the weingarten matrix is given by (2.14) and the fundamental forms are described by (2.18), (2.19) and (2.20). The proof is complete. $\square$

3. Main results. In this section we will present a class of solutions for the n -dimensional generalized Helmholtz and for the n -dimensional Helmholtz equation.

The following result provides a class of solutions for the n -dimensional generalized Helmholtz equation in terms of biharmonic functions and provide necessary and sufficient conditions for a solution of the n -dimensional generalized Helmholtz equation also be a solution for the n -dimensional Helmholtz equation.

Theorem 3.1. *Let $A : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function, a, b non-zero real constants. Then the functions $h(u) = \frac{A(u)}{a + b\|u\|^2}$, $\|u\|^2 \neq -\frac{a}{b}$ and $c = ab$*

- (1) *are solutions for the n -dimensional generalized Helmholtz equation if and only if, A is biharmonic,*
- (2) *are solutions for the n -dimensional Helmholtz equation if and only if, A is biharmonic and satisfy*

$$(3.1) \quad (a + b\|u\|^2)\Delta A - 4b\langle \nabla A, u \rangle + 2b(4 - n)A = 0.$$

Proof. (1) Consider

$$(3.2) \quad h = \frac{A}{T}, \quad \text{where} \quad T = a + b\|u\|^2.$$

Calculating the Laplacian of h we have

$$\Delta h = \frac{\Delta A}{T} + 2 \left\langle \nabla A, \nabla \left(\frac{1}{T} \right) \right\rangle + A \Delta \left(\frac{1}{T} \right).$$

Using the expression of T given in (3.2), we get

$$\begin{aligned} \Delta h &= \frac{\Delta A}{T} - 2 \frac{\langle \nabla A, \nabla T \rangle}{T^2} - A \frac{\Delta T}{T^2} + 2A \frac{\|\nabla T\|^2}{T^3} \\ &= \frac{1}{T^2} (T\Delta A - 4b\langle \nabla A, u \rangle + 2b(4 - n)A - 8abh). \end{aligned}$$

This equation can be written as

$$(3.3) \quad \frac{T^2}{8} \left(\Delta h + \frac{8}{T^2} abh \right) = \frac{1}{8} (T\Delta A - 4b\langle \nabla A, u \rangle + 2b(4 - n)A).$$

Thus, for $\Omega(u) = \frac{2\sqrt{2}}{a + b\|u\|^2}$ and $c = ab$ the function $h = \frac{A}{T}$ is a solution for the n -dimensional generalized Helmholtz equation (1.2) if and only if

$$\Delta (T\Delta A - 4b\langle \nabla A, u \rangle + 2b(4 - n)A) = T\Delta^2 A = 0.$$

Therefore, $h = \frac{A}{a + b\|u\|^2}$ is a solution for the n -dimensional generalized Helmholtz equation if and only if A is biharmonic.

(2) The result it follows from item 1 and (3.3). The proof is complete. $\square$

Proposition 3.2. *The functions $h(u) = \frac{A(u)}{a + b\|u\|^2}$, where $A(u)$ is given by*

$$A(u) = B(u) + \langle u, w(u) \rangle,$$

$w : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $w = (w^1, w^2, \dots, w^n)$, $w^i, B : \mathbb{R}^n \rightarrow \mathbb{R}$ are harmonic functions,

(1) are solutions for the n -dimensional generalized Helmholtz equation,

(2) are solutions for the n -dimensional Helmholtz equation if the following relation is satisfied

$$(3.4) \quad (a + b\|u\|^2)(w_{,1}^1 + w_{,2}^2 + \dots + w_{,n}^n) - 2b \left\langle \nabla B + \sum_{i=1}^n u_i \nabla w^i + \frac{2-n}{2} w, u \right\rangle + (4-n)bB = 0.$$

Proof. (1) It is easy to verify that

$$(3.5) \quad \begin{aligned} \Delta A &= 2(w_{,1}^1 + w_{,2}^2 + \dots + w_{,n}^n), \\ \Delta^2 A &= 2((\Delta w^1)_{,1} + (\Delta w^2)_{,2} + \dots + (\Delta w^n)_{,n}). \end{aligned}$$

By (3.5) and the fact that w^i are harmonic functions, it follows that $A(u)$ is biharmonic and the result follows from item 1 of Theorem 3.1.

(2) Also

$$\nabla A = \nabla B + \sum_{i=1}^n u_i \nabla w^i,$$

using this expression and (3.5) in (3.1) we get (3.4) and the result follows from item 2 of Theorem 3.1. The proof is complete. $\square$

Remark 3.3. Using the Proposition 3.2, we obtain explicit solutions for the n -dimensional generalized Helmholtz equation and for the n -dimensional Helmholtz equation, for this, we will obtain biharmonic functions through harmonic functions.

1. Considering $n = 2k$ and the holomorphic functions $g_k : \mathbb{C} \rightarrow \mathbb{C}$, the function

$$A(u) = \sum_{r=1}^k \langle 1, g_r(z_r) \rangle, \quad u = (z_1, \dots, z_r, \dots, z_k), \quad z_r = (u_{2r-1}, u_{2r}), \quad r = 1, \dots, k,$$

is harmonic.

We note that these functions satisfy (3.4) (here $w = (0, \dots, 0)$) if

(1) for $n = 2(k = 1)$,

$$g_1(z_1) = c_1 z_1 + i b_1,$$

(2) for $n = 4(k = 2)$,

$$g_r(z_r) = a_r \ln z_r + b_r, \quad r = 1, 2, \text{ with the condition } \sum_{r=1}^2 a_r = 0,$$

(3) for $n \neq 2, n \neq 4$,

$$g_r(z_r) = \frac{2a_r z_r^{2-\frac{n}{2}}}{4-n} + b_r, \quad r = 1, \dots, k \text{ with the condition } \sum_{r=1}^k b_r = 0.$$

2. Considering $n = 2k + 1$ and the holomorphic functions $g_k : \mathbb{C} \rightarrow \mathbb{C}$, the function

$$A(u) = \sum_{r=1}^k f_r(u_{2k+1}) \langle 1, g_r(z_r) \rangle + f_{2k+1}(u_{2k+1}),$$

where $f_r(u_{2k+1}) = a_r u_{2k+1} + s_r$, $f_{2k+1}(u_{2k+1}) = a_{2k+1} u_{2k+1} + s_{2k+1}$, $u = (z_1, \dots, z_r, \dots, z_k, u_{2k+1})$, $z_r = (u_{2r-1}, u_{2r})$, $r = 1, \dots, k$, is harmonic.

We note that these functions satisfy (3.4) (here $w = (0, \dots, 0, a_{2k+1})$) if

(1) for $n = 3(k = 1)$,

$$g_1(z_1) = c_1 \ln z_1 - \frac{a_3}{a_1}, \quad f_1(u_3) = a_1 u_3, \quad f_3(u_3) = a_3 u_3,$$

(2) for $n = 5(k = 2)$,

$$g_r(z_r) = c_r \ln z_r + b_r, \quad f_r(u_5) = a_r u_5, \quad r = 1, 2, \quad f_5(u_5) = a_5 u_5 \text{ with the condition}$$

$$\sum_{r=1}^2 c_r a_r = 0,$$

(3) for $n \neq 5$,

$$g_r(z_r) = \frac{2c_r z_r^{\frac{5-n}{2}}}{5-n} + b_r, \quad f_r(u_{2k+1}) = a_r u_{2k+1}, \quad r = 1, \dots, k, \quad f_{2k+1}(u_{2k+1}) = a_{2k+1} u_{2k+1} \text{ with the condition}$$

$$\sum_{r=1}^k b_r a_r + a_{2k+1} = 0.$$

In the following result we provide radial solutions for the n -dimensional generalized Helmholtz equation.

Theorem 3.4. *The functions $h(u) = \frac{A(u)}{a + b\|u\|^2}$ are radial solutions for the n -dimensional generalized Helmholtz equation if, and only if, $A(u)$ is a radial biharmonic function. Moreover,*

$$(3.6) \quad h(u) = \begin{cases} \frac{(c_1\|u\|^2 - c_2) \ln \|u\|^2 + c_3\|u\|^2 + c_4}{a + b\|u\|^2}, & \text{if } n = 2, \\ \frac{2c_4\|u\|^4 - 2c_1\|u\|^2 \ln \|u\|^2 + 2c_3\|u\|^2 + c_2}{2\|u\|^2(a + b\|u\|^2)}, & \text{if } n = 4, \\ \frac{4\|u\|^{2-n}((n-4)c_1 + nc_2\|u\|^2)}{n(n^2 - 6n + 8)(a + b\|u\|^2)} + \frac{c_3\|u\|^2 + c_4}{a + b\|u\|^2}, & \text{if } n \neq 2, n \neq 4, \end{cases}$$

where c_i are real constants.

Proof. According to Theorem 3.1, $h(u) = \frac{A(u)}{a + b\|u\|^2}$ is a solution of the n -dimensional generalized Helmholtz equation if, and only if, $A(u)$ is a biharmonic function.

On the other hand, h is a radial function if, and only if, $A(u)$ is a radial function i.e. $A(u) = J(t)$, $t = \|u\|^2$, $u \in U$ for some differentiable function J .

Calculating the bilaplacian of A we obtain

$$\Delta^2 A = 16t^2 J^{(4)}(t) + 16(n+2)tJ'''(t) + 4n(n+2)J''(t).$$

Using the fact that A is biharmonic we get

$$(3.7) \quad 4t^2 J^{(4)}(t) + 4(n+2)tJ'''(t) + n(n+2)J''(t) = 0.$$

The solutions of (3.7) are given by

for $n = 2$

$$J(t) = (c_1 t - c_2) \ln t + c_3 t + c_4,$$

for $n = 4$

$$J(t) = \frac{c_2}{2t} + c_4 t - c_1 \ln t + c_3,$$

for $n \neq 2$ and $n \neq 4$

$$J(t) = \frac{4t^{1-\frac{n}{2}}((n-4)c_1 + ntc_2)}{n(n^2 - 6n + 8)} + c_3 t + c_4.$$

The proof is complete. $\square$

Corollary 3.5. *The radial solutions of the n -dimensional generalized Helmholtz equation given in (3.6) are also solutions for the n -dimensional Helmholtz equation if:*

$$A(u) = (\|u\|^2 - a) \left(\frac{c_2}{a} \ln \|u\|^2 + c_3 \right) - 4c_2, \quad \text{if } n = 2,$$

$$A(u) = \frac{c_2}{2\|u\|^2} \left(1 - \frac{\|u\|^4}{a^2} - \frac{2\|u\|^2}{a} \ln \|u\|^2 \right) + c_3, \quad \text{if } n = 4,$$

$$A(u) = \frac{4c_2\|u\|^{2-n}(a(n-4) + n\|u\|^2)}{n(n^2 - 6n + 8)} + \frac{c_3}{n-4}((n-4)\|u\|^2 + an), \quad \text{if } n \neq 2, n \neq 4.$$

Proof. By Theorem 3.1, the solutions of the n -dimensional generalized Helmholtz equation are also solutions for the n -dimensional Helmholtz equation if the function A satisfies (3.1).

Thus, by Theorem 3.4, for $n = 2$,

$$A(u) = (c_1\|u\|^2 - c_2) \ln \|u\|^2 + c_3\|u\|^2 + c_4.$$

Differentiating we obtain

$$A_{,i} = 2u_i \left(c_1 \ln \|u\|^2 + \frac{c_1\|u\|^2 - c_2}{\|u\|^2} + c_3 \right), \quad i = 1, 2, \quad \Delta A = 4c_1 \ln \|u\|^2 + 8c_1 + 4c_3.$$

Using these expressions in (3.1), we get

$$(ac_1 - c_2) \ln \|u\|^2 + 2ac_1 + ac_3 + 2c_2 + c_4 = 0.$$

Hence, it follows the result.

Similarly, from Theorem 3.4, for $n = 4$,

$$A(u) = \frac{c_2}{2\|u\|^2} + c_4\|u\|^2 - c_1 \ln \|u\|^2 + c_3.$$

Differentiating we obtain

$$A_{,i} = 2u_i \left(-\frac{c_2}{2\|u\|^4} + c_4 - \frac{c_1}{\|u\|^2} \right), \quad i = 1, \dots, 4, \quad \Delta A = -\frac{4c_1}{\|u\|^2} + 8c_4.$$

Using these expressions in (3.1),

$$c_1 + 2ac_4 + \frac{c_2 - ac_1}{\|u\|^2} = 0.$$

Hence, it follows the result.

Finally, by Theorem 3.4, for $n \neq 2, n \neq 4$,

$$A(u) = \frac{4\|u\|^{2-n}((n-4)c_1 + n\|u\|^2c_2)}{n(n^2 - 6n + 8)} + c_3\|u\|^2 + c_4.$$

Differentiating we obtain

$$\begin{aligned}
 A_{,i} &= 2u_i \left(\frac{(4-2n)\|u\|^{-n}((n-4)c_1 + c_2n\|u\|^2) + 4c_2n\|u\|^{2-n}}{n(n^2-6n+8)} + c_3 \right), \\
 A_{,ii} &= 2 \left(\frac{(4-2n)\|u\|^{-n}((n-4)c_1 + c_2n\|u\|^2) + 4c_2n\|u\|^{2-n}}{n(n^2-6n+8)} + c_3 \right) \\
 &\quad + 4u_i^2 \left(\frac{-n(2-n)\|u\|^{-(n+2)}((n-4)c_1 + c_2n\|u\|^2) + n(8-4n)c_2\|u\|^{-n}}{n(n^2-6n+8)} \right), \\
 i &= 1, \dots, n.
 \end{aligned}$$

Using these expressions in (3.1),

$$\frac{4t^{1-\frac{n}{2}}(c_1 - ac_2)}{n-2} + ac_3n - c_4(n-4) = 0.$$

From this expression it follows the result. The proof is complete. $\square$

4. Classes of generalized Weingarten hypersurfaces. In this section we introduce two classes of generalized Weingarten hypersurfaces obtained by a congruence of spheres. These hypersurfaces are associated with solutions of the equations (1.1) and (1.2).

Definition 4.1. *An orientable hypersurface $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, is called generalized Weingarten hypersurface depending on support function and the radius function (in short, RSGW-hypersurface), if the r th-mean curvatures H_{n-1}, H_n , the support function Ψ with relation to origin, the radius function R with respect to hyperplane $\Pi = \{(u_1, u_2, \dots, u_n, 0)/u_i \in \mathbb{R}\}$, satisfy*

$$(4.1) \quad n \frac{H_{n-1}}{H_n} - nR - 2\Psi = 0.$$

Remark 4.2. We observe that this class of hypersurfaces satisfies the geometric property that the set spheres with center $p + R(p)N(p)$ and radius $R(p) = \frac{H_{n-1}(p)}{H_n(p)} - \frac{2}{n}\Psi(p)$ are tangent to hyperplane $\Pi = \{(u_1, u_2, \dots, u_n, 0)/u_i \in \mathbb{R}\}$.

Definition 4.3. *An orientable hypersurface $M \subset \mathbb{R}^{n+1}$, $n \geq 2$, is called harmonic generalized Weingarten hypersurface depending on support function and the radius function (in short RSHGW-hypersurface) if the r th-mean curvatures H_{n-1}, H_n , the support function Ψ with relation to origin, the radius function R with respect to hyperplane $\Pi = \{(u_1, u_2, \dots, u_n, 0)/u_i \in \mathbb{R}\}$ and the projection*

function d of M on Π , satisfy

$$(4.2) \quad \Delta \left(\frac{R}{d} \left(n \frac{H_{n-1}}{H_n} - nR - 2\Psi \right) \right) = 0.$$

Theorem 4.4. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K \neq 0$ as in Theorem 2.11.*

(1) *M is a RSGW-hypersurface if, and only if*

$$\Delta_L h + \frac{8}{T^2} h = 0,$$

(2) *M is a RSHGW-hypersurface if, and only if*

$$\Delta \left(\frac{T^2}{8} \left(\Delta_L h + \frac{8}{T^2} h \right) \right) = 0,$$

where Δ_L is the Laplacian with respect to the metric $L = (L_i)$ induced by Y , i.e. $L_i = \langle Y_{,i}, Y_{,i} \rangle$ and $T = 1 + \|Y\|^2$.

Proof. From (2.11) and (2.13), we have that $\Psi = \frac{2h}{T}$, also from (2.14) and (2.15) the Weingarten matrix W of X is given by $W = 2(TV - 2RI)^{-1}$ where $V = (V_{,ij})$.

Since, V is simetric, let $\{a_1, \dots, a_n\}$ the eigenvalues of V . It follows that $\lambda_i = \frac{2}{Ta_i - 2R}$, $1 \leq i \leq n$, are the eigenvalues of W and consequently, the principal curvatures $k_1, \dots, k_n$ of M are given by

$$(4.3) \quad k_i = -\lambda_i = \frac{2}{2R - Ta_i}, \quad 1 \leq i \leq n.$$

Using (4.3) and the definition 2.5, we get that

$$\begin{aligned} \frac{H_{n-1}}{H_n} &= \frac{1}{n} \sum_{i=1}^n \frac{1}{k_i} = \frac{1}{n} \sum_{i=1}^n \left(R - \frac{T}{2} a_i \right) = \frac{1}{n} \left(nR - \frac{T}{2} \sum_{i=1}^n a_i \right) \\ &= R - \frac{T}{2n} \text{tr}(V) = R - \frac{T}{2n} \Delta_L h. \end{aligned}$$

Now using this expression we get

$$(4.4) \quad n \frac{H_{n-1}}{H_n} - nR - 2\Psi = -\frac{T}{2} \left(\Delta_L h + \frac{8}{T^2} h \right).$$

From (2.11) the projection d of M on the hyperplane Π is given by $d = -\frac{2R}{T}$.

Thus

$$(4.5) \quad \frac{R}{d} \left(n \frac{H_{n-1}}{H_n} - nR - 2\Psi \right) = \frac{T^2}{4} \left(\Delta_L h + \frac{8}{T^2} h \right).$$

From (4.4) and (4.5) it follow the result. The proof is complete. $\square$

Theorem 4.5. *Let M be an orientable hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K \neq 0$ as in Theorem 2.11, with $Y = (u, 0)$ and $h(u) = \frac{A(u)}{1 + \|u\|^2}$ where $A : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function. M is a RSHGW-hypersurface if, and only if, $A(u)$ is biharmonic.*

Proof. Since $Y = (u, 0)$ we get that $T = 1 + \|u\|^2$ and $\Delta_L = \Delta$. From Theorem 4.4, it follows

$$(4.6) \quad \frac{R}{d} \left(n \frac{H_{n-1}}{H_n} - nR - 2\Psi \right) = \frac{T^2}{4} \left(\Delta h + \frac{8}{T^2} h \right).$$

From Theorem 3.1, for $a = b = 1$ and $\Omega(u) = \frac{2\sqrt{2}}{1 + \|u\|^2}$, the function $h(u) = \frac{A(u)}{1 + \|u\|^2}$ is a solution of the n -dimensional generalized Helmholtz equation (1.2) if, and only if, $A(u)$ is biharmonic, i.e. from (4.6)

$$\Delta \left(\frac{R}{d} \left(n \frac{H_{n-1}}{H_n} - nR - 2\Psi \right) \right) = 0 \iff A \text{ is biharmonic.}$$

Thus, the proof is complete. $\square$

Corollary 4.6. *Let M be a RSHGW-hypersurface in $\mathbb{R}^{n+1}$ with Gauss-Kronecker curvature $K \neq 0$ as in Theorem 2.11, with $Y = (u, 0)$ and $h(u) = \frac{A(u)}{1 + \|u\|^2}$ where $A : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function. M is a SRGW-hypersurface if, and only if, $A(u)$ is biharmonic and satisfies (3.1).*

Proof. From Theorem 4.5, $A(u)$ is biharmonic and from (4.4), M is a SRGW-hypersurface if, and only if, $\Delta h + \frac{8}{T^2} h = 0$, which is equivalent to (3.1), for $a = b = 1$. The proof is complete. $\square$

The following result classify the RSHGW-hypersurfaces of rotation.

Theorem 4.7. *A hypersurface M given as in Theorem 4.5 is RSHGW-hypersurface of rotation if and only if h is given by (3.6) for $a = b = 1$.*

Proof. We claim that M is of rotation if, and only if, $h(u)$ is a radial function.

In fact, suppose that M is a hypersurface of rotation and X is a parametrization of M given by Theorem 4.5. Without loss of generality, we can suppose that the axis of rotation is the axis x_{n+1} . In this way, the sections orthogonal to the axis x_{n+1} determine in $X(U)$ spheres $(n-1)$ -dimensional centered in axis x_{n+1} .

Since, Y is the inclusion map, the Gauss map is the inverse of the stereographic projection which ensures that the immersion X takes $(n-1)$ -dimensional spheres centered in the origin in $(n-1)$ -dimensional spheres centered on the axis of rotation. Note also that on these spheres the angle between X and N is constant, this is, $\langle X, N \rangle = k$, where k is a constant.

On the other hand, from Theorem 2.11, we get

$$\begin{aligned} k &= \langle X, N \rangle = \left\langle \left(Q - \frac{2R}{1 + \|u\|^2} u, -\frac{2R}{1 + \|u\|^2} \right), \frac{1}{1 + \|u\|^2} (2u, 1 - \|u\|^2) \right\rangle \\ &= \frac{2}{(1 + \|u\|^2)^2} (R + h)(1 + \|u\|^2) - \frac{2R}{(1 + \|u\|^2)^2} \|u\|^2 - \frac{2R}{(1 + \|u\|^2)^2} \\ &= \frac{2h}{1 + \|u\|^2}. \end{aligned}$$

Therefore, $h = J(\|u\|^2)$, where $J(\|u\|^2) = \frac{k(1 + \|u\|^2)}{2}$, for some differentiable function J , consequently h is a radial function.

Conversely, suppose that $h : U \rightarrow \mathbb{R}$ is a radial function, i.e., $h(u) = J(\|u\|^2)$, $u = (u_1, \dots, u_n) \in U$, for some differentiable function J .

Denoting by $t = \|u\|^2$ we get that $h_{,i} = 2u_i J'$ and consequently $\nabla h = 2J' u$.

Since, $L_{ii} = \langle Y_{,i}, Y_{,i} \rangle = 1$, $1 \leq i \leq n$, we obtain

$$Q = \nabla h = 2J' u, \quad R = \langle \nabla h, u \rangle - h = 2J' t - J \quad \text{and} \quad T = 1 + \|u\|^2 = 1 + t.$$

Using these expressions in (2.11)

$$(4.7) \quad X = \left(\left(2J' - 2 \frac{(2J' t - J)}{1 + t} \right) u, -2 \frac{(2J' t - J)}{1 + t} \right)$$

Thus, if $-2 \frac{(2J' t - J)}{1 + t} = q$ for constant on $\|u\|^2$, we get

$$(4.8) \quad \left\| \left(2J' - 2 \frac{(2J' t - J)}{1 + t} \right) u \right\|^2 = (2J' + q)^2 t.$$

Therefore, the orthogonal sections to axis x_{n+1} determine in $X(U)$ spheres $(n-1)$ -dimensional centered in axis x_{n+1} .

On the other hand, since $h(u) = \frac{A(u)}{1 + \|u\|^2}$, we have that h is a radial function if, and only if, $A(u)$ is a radial function i.e. $A(u) = J(t)$, $t = \|u\|^2$, $u \in U$ for some differentiable function J and the result it follows from Theorem 3.4, for $a = b = 1$. The proof is complete. $\square$

5. RSHGW-surfaces and RSGW-surfaces. In this section we will study the RSGW-surfaces and the RSHGW-surfaces.

For $n = 2$, from Definitions 4.1 and 4.3, it follows that the RSGW-surfaces and the RSHGW-surfaces satisfy respectively

$$(5.1) \quad \frac{H}{K} - R - \Psi = 0,$$

$$(5.2) \quad \Delta \left(\frac{R}{d} \left(\frac{H}{K} - R - \Psi \right) \right) = 0,$$

where K is the Gaussian curvature and H is the mean curvature of M .

Corollary 5.1. *A surface given as in Theorem 4.4 with $Y(u) = (g(u), 0)$ and $h(u) = \frac{\langle 1, B \rangle + \langle g, C \rangle}{1 + \|g\|^2}$ where $g, B, C : U \subset \mathbb{C} \rightarrow \mathbb{C}$ are holomorphic functions is a RSHGW-surface.*

Proof. For $Y = (g(u), 0)$, we get $T = 1 + \|g\|^2$ and $\Delta_L h = \frac{1}{\|g'\|^2} \Delta h$. From Theorem 4.4,

$$(5.3) \quad \Delta \left(\frac{T^2}{8} \left(\Delta_L h + \frac{8}{T^2} h \right) \right) = \Delta \left(\frac{1}{\frac{8\|g'\|^2}{T^2}} \left(\Delta h + \frac{8\|g'\|^2}{T^2} h \right) \right).$$

From Theorem 1 in [20], for $c = 1$ and $\Omega(u) = \frac{2\sqrt{2}\|g'\|}{1 + \|g\|^2}$, the function $h = \frac{\langle 1, B \rangle + \langle g, C \rangle}{1 + \|g\|^2}$ is a solution for the two-dimensional generalized Helmholtz equation (1.2).

From (5.3) and Theorem 4.4, follows the result. The proof is complete. $\square$

Corollary 5.2. *A surface given as in Theorem 4.4 with $Y(u) = (g(u), 0)$ and $h = \frac{\langle 1, B \rangle + \langle g, C \rangle}{1 + \|g\|^2}$, where g, B, C are holomorphic functions, $C = \int (B'g - Bg' + ic_1g')dz$ and c_1 is a real constant, is a SRMC-surface.*

Proof. Similarly to the proof of Corollary 5.1, from Theorem 4.4, we obtain

$$(5.4) \quad \Delta_L h + \frac{8}{T^2} h = \frac{1}{|g'|^2} \left(\Delta h + \frac{8\|g'\|^2}{T^2} h \right).$$

From Corollary 1 in [20], for $c = 1$ and $\Omega(u) = \frac{2\sqrt{2}\|g'\|}{1 + \|g\|^2}$, the function $h = \frac{\langle 1, B \rangle + \langle g, C \rangle}{1 + \|g\|^2}$ is a solution for the two-dimensional Helmholtz equation (1.1). Thus, from (5.4) and Theorem 4.4, it follows the result. The proof is complete. $\square$

Remark 5.3. From Theorem 2.11, Corollaries 5.1 and 5.2, we get a Weierstrass type representation for these surfaces which depend of three holomorphic functions given by

$$(5.5) \quad X(u) = \left(\frac{g' \nabla h}{\|g'\|^2} - \frac{2R}{T}g, -\frac{2R}{T} \right).$$

The RSHGW-surfaces and the RSGW-surfaces generically have singularities given by the expression (2.17).

Example 1. Considering $g(u) = u$, $B(u) = e^u$, $C(u) = \cos u$ in the Corollary 5.1, we obtain a RSHGW-surface given in the Fig. 1.

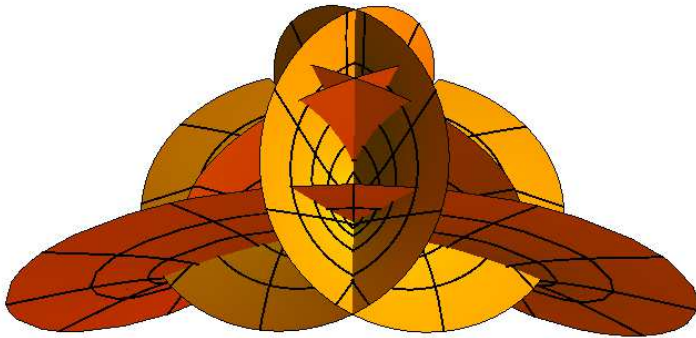


Fig. 1. RSHGW-surface

Example 2. Considering $g(u) = u^2$, $B(u) = e^u$, $C(u) = u^3$ in the Corollary 5.1, we obtain a RSHGW-surface given in the Fig. 2.

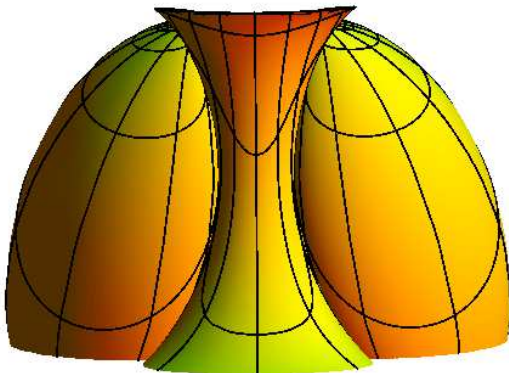


Fig. 2. RSHGW-surface

Example 3. Considering $g(u) = \sinh u$, $B(u) = \cosh u$, $C(u) = u^2$ in the Corollary 5.1, we obtain a RSHGW-surface given in the Fig. 3.

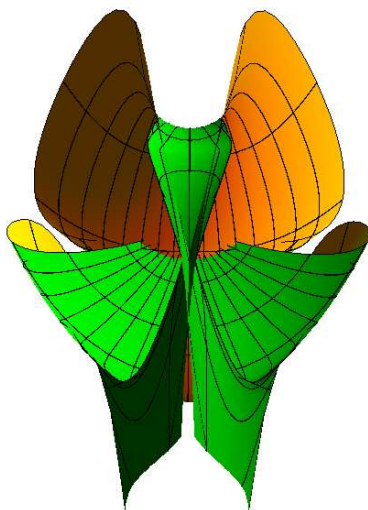


Fig. 3. RSHGW-surface

Example 4. Considering $g(u) = u$, $B(u) = \cosh u$, $C(u) = u \cosh u - 2 \sinh u + ic_1 u$ in the Corollary 5.2, we obtain a RSGW-surface given in the Fig. 4.

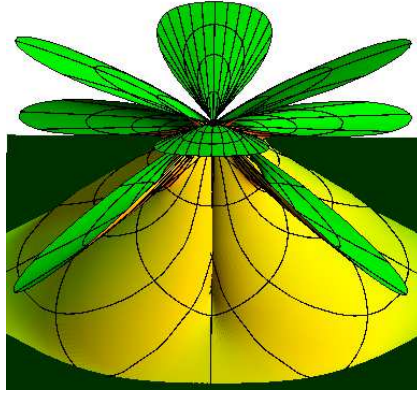


Fig. 4. RSGW-surface

Example 5. Considering $g(u) = u + e^u$, $B(u) = u^2$, $C(u) = \frac{u^3}{3} - (u^2 - 4u + 4)e^u + ic_1(u + e^u)$ in the Corollary 5.2, we obtain a RSGW-surface given in the Fig. 5.

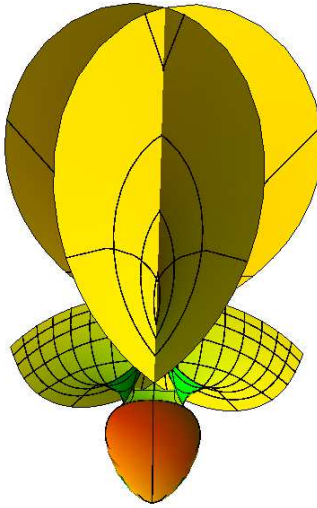


Fig. 5. RSGW-surface

Example 6. Considering $g(u) = \sin u$, $B(u) = u^3$, $C(u) = (12 - 6u^2) \cos u + (ic_1 + 12u - u^3) \sin u$ in the Corollary 5.2, we obtain a RSGW-surface given in the Fig. 6.

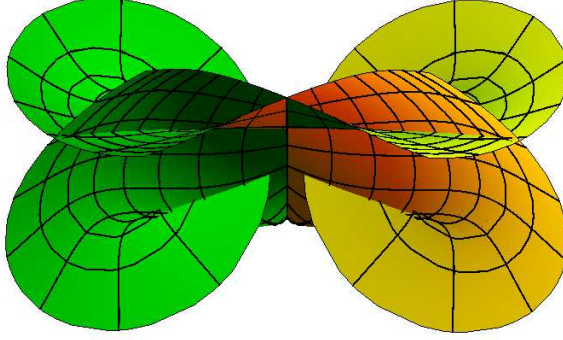


Fig. 6. RSGW-surface

The following result classify the RSHGW-surfaces of rotation.

Theorem 5.4. A surface given as in Corollary 5.1 is RSHGW-surface of rotation if and only if $g(u) = e^u$ and $h = \frac{a_2 + c_1 u_1 + e^{2u_1}(a_3 + c_2 u_1)}{1 + e^{2u_1}}$. The RSHGW-surface of rotation X can be locally parametrized by

$$(5.6) \quad X(u) = (M_1(u_1) \cos u_2, M_1(u_1) \sin u_2, N_1(u_1)),$$

where

$$M_1(u_1) = \frac{c_1 + e^{4u_1}(4a_2 - c_1 + 4c_1 u_1 + 4(a_3 + c_2 u_1) \cosh(2u_1) - 2c_2 \sinh(2u_1))}{e^{u_1} (1 + e^{2u_1})^3},$$

$$N_1(u_1) = \frac{2(a_2 - c_1 + e^{4u_1}(a_3 + c_2(u_1 - 1)) + c_1 u_1 - e^{2u_1}(c_3 + (c_2 - 3c_1)u_1))}{(1 + e^{2u_1})^3},$$

$$c_3 = a_3 - 3a_2 + c_1 + c_2.$$

Proof. From Theorem 4.7, X is of rotation if, and only if, $g(w) = w$ and $h(w) = J(|w|^2)$, $w \in \mathbb{C}$, for some differentiable function J .

Making the variable change

$$g(w) = e^u, u = u_1 + iu_2 \in \mathbb{C},$$

we get $g(u) = e^u$, $h(u) = J(e^{2u_1})$, thus, we have that $h_{,2} = 0$.

From Corollary 5.1, we have

$$(5.7) \quad h = \frac{\langle 1, B \rangle + \langle e^u, C \rangle}{1 + e^{2u_1}}.$$

Differentiating (5.7) with relation to u_2 and using the fact that $h_{,2} = 0$ we obtain

$$\langle 1, iB' \rangle + \langle e^u, i(C' - C) \rangle = 0.$$

Using Lemma 2.2, we get

$$iB' = -\overline{z_1}e^u + ic_1, \quad i(C' - C) = ic_2e^u + z_1,$$

whose solutions are given by

$$B = i\overline{z_1}e^u + c_1u + z_2, \quad C = (z_3 + c_2u)e^u + iz_1,$$

where $c_1, c_2 \in \mathbb{R}$, $z_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$, $k = 1, 2, 3$.

Thus, we obtain the function h given above.

On the other hand,

$$\begin{aligned} \frac{g'}{\|g'\|^2} \nabla h &= \frac{c_1 (1 + e^{2u_1}(1 - 2u_1)) + e^{2u_1} (2a_3 - 2a_2 + c_2 + c_2 e^{2u_1} + 2c_2 u_1)}{e^{u_1} (1 + e^{2u_1})^2} \\ &\quad \times (\cos u_2, \sin u_2, 0), \\ (5.8) \quad R &= -\frac{(1 + e^{2u_1})N_1(u_1)}{2}, \quad T = 1 + e^{2u_1}. \end{aligned}$$

From equations (5.5) and (5.8) we get (5.6). $\square$

Example 7. Considering $a_2 = -1$, $a_3 = 0$, $c_1 = 1$, $c_2 = 0$ in the Theorem 5.4, we obtain a RSHGW-surface of rotation (see Fig. 8) with profile curve given in the Fig. 7. Moreover, this surface has one isolated singularity, this singularity is generated by the intersection of the profile curve with the axis of rotation.

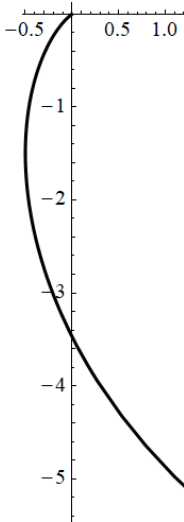


Fig. 7. Profile curve

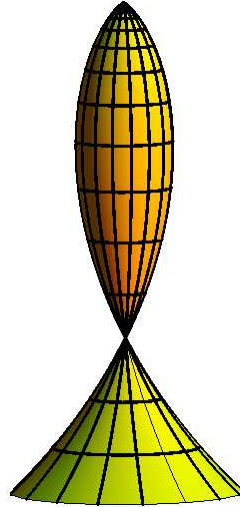


Fig. 8. RSHGW-surface of rotation

Example 8. Considering $a_2 = -1$, $a_3 = 2$, $c_1 = 1$, $c_2 = 3$ in the Theorem 5.4, we obtain a RSHGW-surface of rotation (see Fig. 10) with profile curve given in the Fig. 9. Moreover, this surface has a circle of singularities, these singularities are generated by a point on the profile curve where it is not regular.

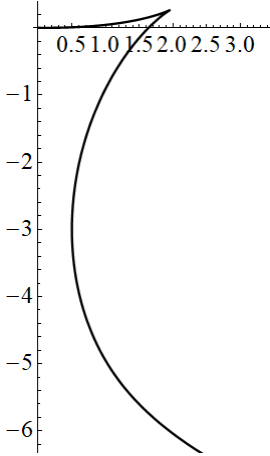


Fig. 9. Profile curve

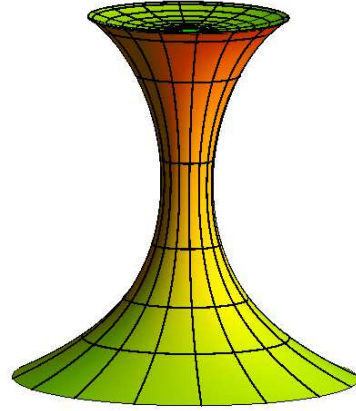


Fig. 10. RSHGW-surface of rotation

Example 9. Considering $a_2 = 1$, $a_3 = -2$, $c_1 = 1$, $c_2 = \frac{1}{3}$ in the Theorem 5.4, we obtain a RSHGW-surface of rotation (see Fig. 12) with profile curve given in the Fig. 11.

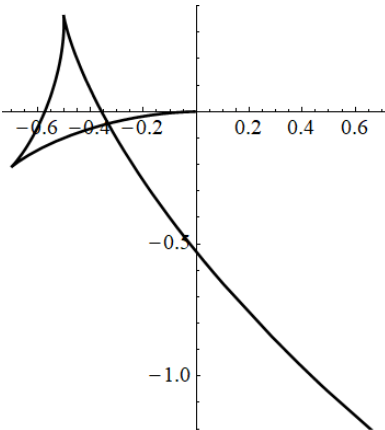


Fig. 11. Profile curve

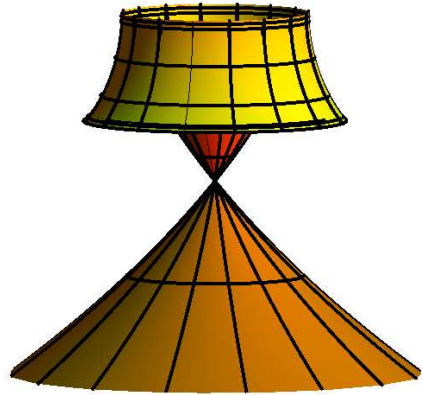


Fig. 12. RSHGW-surface of rotation

Example 10. Considering $a_2 = \frac{1}{2}$, $a_3 = \frac{2}{3}$, $c_1 = 0$, $c_2 = 1$ in the Theorem 5.4, we obtain a RSHGW-surface of rotation (see Fig. 14) with profile curve given in the Fig. 13.

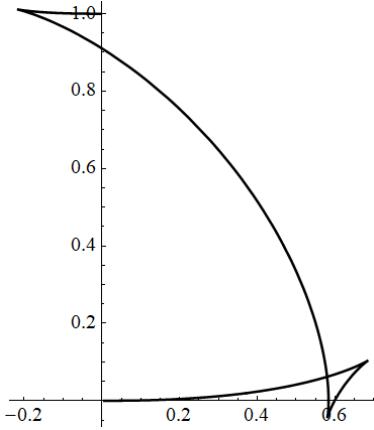


Fig. 13. Profile curve

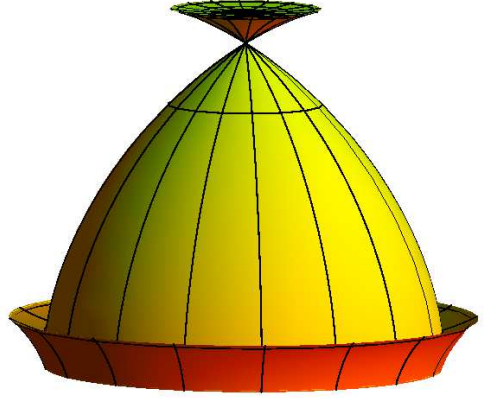


Fig. 14. RSHGW-surface of rotation

The following result classify the RSGW-surfaces of rotation.

Theorem 5.5. *A surface given as in Corollary 5.2 is a RSGW-surface of rotation if and only if $g(u) = e^u$ and $h = c_1 - (a_2 + c_1(u_1 - 1)) \tanh(u_1)$. The RSGW-surface of rotation X can be locally parametrized by*

$$(5.9) \quad X(u) = (M_2(u_1) \cos u_2, M_2(u_1) \sin u_2, N_2(u_1)),$$

where

$$\begin{aligned} M_2(u_1) &= c_1 \cosh(u_1) - ((a_2 + c_1(u_1 - 1)) \operatorname{sech}(u_1) + c_1 \sinh(u_1)) \tanh^2(u_1), \\ N_2(u_1) &= \frac{2(c_1 + (a_2 + c_1(u_1 - 1)) \operatorname{sech}^2(u_1) - (a_2 + c_1(u_1 - 2)) \tanh(u_1))}{1 + e^{2u_1}}. \end{aligned}$$

Proof. Similar to proof of Theorem 5.4, from Theorem 4.7, X is of rotation if, and only if, $g(w) = w$ and $h(w) = J(\|w\|^2)$, $w \in \mathbb{C}$, for some differentiable function J . Making the variable change

$$g(w) = e^u, u = u_1 + iu_2 \in \mathbb{C},$$

we obtain $g(u) = e^u$, $h(u) = J(e^{2u_1})$, thus, we have that $h_{,2} = 0$.

From Corollary 5.2, we get

$$(5.10) \quad h = \frac{\langle 1, B \rangle + \langle e^u, C \rangle}{1 + e^{2u_1}}.$$

Differentiating (5.10) with relation to u_2 , using the fact that $h_{,2} = 0$ and Lemma 2.2, we get

$$B = i\bar{z}_1 e^u + c_1 u + z_2, \quad C = (z_3 + c_2 u) e^u + i z_1,$$

where $c_1, c_2 \in \mathbb{R}, z_k = a_k + i b_k, a_k, b_k \in \mathbb{R}, k = 1, 2, 3$.

Again using the Corollary 5.2,

$$(5.11) \quad B = c_1 u + z_2, \quad C = (2c_1 - z_2 + i c) e^u - c_1 u e^u.$$

From equations ((5.11) in (5.10) we get the function h given above.

On the other hand,

$$\begin{aligned} \frac{g'}{\|g'\|^2} \nabla h &= e^{-u_1} \left(-(a_2 + c_1(u_1 - 1)) \operatorname{sech}^2(u_1) - c_1 \tanh(u_1) \right) (\cos u_2, \sin u_2, 0), \\ (5.12) \quad R &= -\frac{(1 + e^{2u_1}) N_2(u_1)}{2}, \quad T = 1 + e^{2u_1}. \end{aligned}$$

From equations (5.5) and (5.12) we get (5.9). $\square$

Example 11. Considering $a_2 = 1, c_1 = 0$ in the Theorem 5.5, we obtain a RSGW-surface of rotation (see Fig. 16) with profile curve given in the Fig. 15. Moreover, this surface has one isolated singularity and two circles of singularities, these singularities are generated by the intersection of the profile curve with the axis of rotation and by two point on the profile curve where it is not regular, respectively.

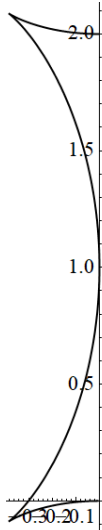


Fig. 15. Profile curve

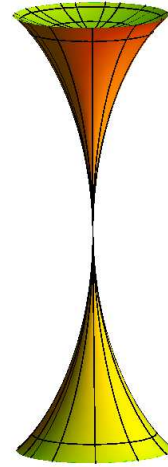


Fig. 16. RSGW-surface of rotation

Example 12. Considering $a_2 = \sqrt{2} + \pi$, $c_1 = \sqrt{3}$ in the Theorem 5.5, we obtain a RSGW-surface of rotation (see Fig. 18) with profile curve given in the Fig. 17. Moreover, this surface has one isolated singularity and two circles of singularities, these singularities are generated by the intersection of the profile curve with the axis of rotation and by two point on the profile curve where it is not regular, respectively.

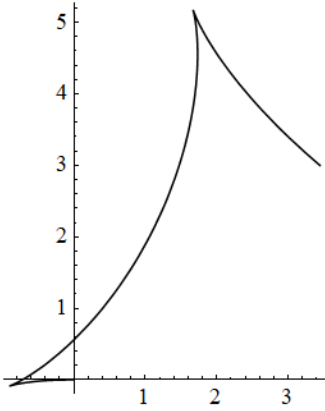


Fig. 17. Profile curve

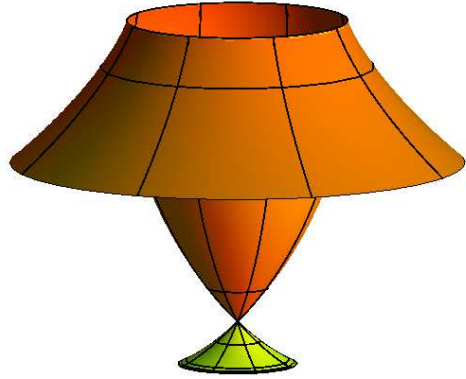


Fig. 17. RSGW-surface of rotation

Example 13. Considering $a_2 = 9$, $c_1 = \frac{1}{2}$ in the Theorem 5.5, we obtain a RSGW-surface of rotation (see Fig. 20) with profile curve given in the Fig. 19. Moreover, this surface has three isolated singularities and two circles of singularities, these singularities are generated by the intersection of the profile curve with the axis of rotation and by two point on the profile curve where it is not regular, respectively.

Example 14. Considering $a_2 = 1$, $c_1 = \frac{1}{7}$ in the Theorem 5.5, we obtain a SRMC-surface of rotation (see Fig. 22) with profile curve given in the Fig. 21. Moreover, this surface has two isolated singularities and one circle of singularities, these singularities are generated by the intersection of the profile curve with the axis of rotation and by a point on the profile curve where it is not regular, respectively.

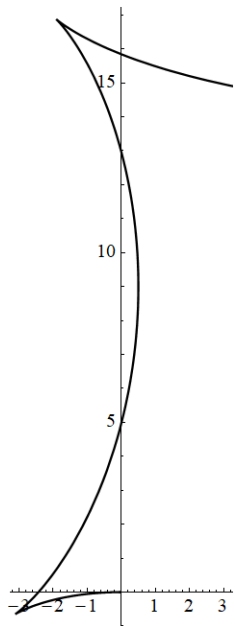


Fig. 19. Profile curve

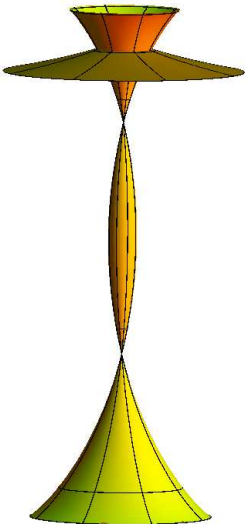


Fig. 20. RSGW-surface of rotation

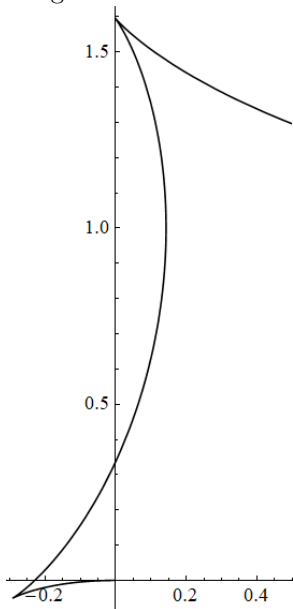


Fig. 21. Profile curve

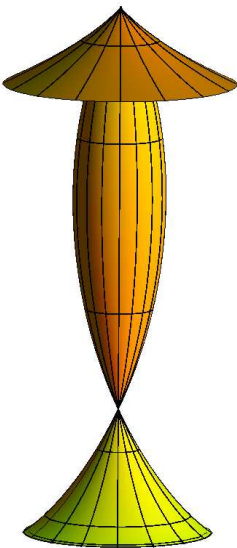


Fig. 22. RSGW-surface of rotation

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