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POWER SERIES AND POSITIVE LINEAR OPERATORS IN WEIGHTED SPACES

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Abstract. A general family of positive linear operators associated with a power expansion is studied. Sufficient conditions are provided to verify that the sequence of operators are an approximation process in weighted spaces with polynomial weights. Examples of applications are included.

1. Introduction. Throughout the work we denote by $\mathbb{P}_n$ the spaces of all algebraic polynomials of degree non greater than n. For each $j \in \mathbb{N}_0$, we use the notations

$$e_j(x) = x^j.$$

Moreover, $I = [0, \infty)$.

Let us begin with a general approach to construct sequences of positive linear operators (see [1], [11] and [12]).

For fixed sequence $\{a_{n,k}\}_{k=0}^{\infty}$ of positive real numbers and $x \in I$, set

$$g_n(x) = \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} x^k, \qquad \text{where} \quad \limsup_{k\to\infty} \left(\frac{a_{n,k}}{k!}\right)^{1/k} = 0. \tag{1}$$

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Here we consider the operators

$$L_n(f,x) = \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} f\Big(\frac{k}{\beta(n)}\Big) x^k, \tag{2}$$

where $\beta(n) \geq 1$ and $\lim_{n\to\infty} \beta(n) = \infty$. Notice that, if $g_n(x) = e^{-nx}$ and $\beta(n) = n$ we obtain the operators of Szász-Mirakyan (see [16]), here denoted by S_n.

For the definition of L_n it is sufficient that $a_{n,0} \neq 0$, but in the proof of Theorem 2 we need that $g_n^{(i)}(x) > 0$, for all $i \in \mathbb{N}$ and $x \in I$. That is the reason why we assume that $a_{n,k} > 0$ for all k. Moreover, we don't consider the case when g_n is a polynomial. We remak that the operators L_n are related with other ones studied by Mortici in [11].

Let $\mathcal{D}(L)$ be the family of all functions $f \in C(I)$ such that, for each $n \in \mathbb{N}$, the series $L_n(f)$ converges absolutely.

We will show that the approximation properties of the operators L_n are related with the the behaviour of $g_n^{(i)}(x)/(\beta^i(n) g_n(x))$. Hence we will assume two different kind of asymptotic: (1) for fixed $r \in \mathbb{N}$ and $x > 0$,

$$\lim_{n\to\infty} \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} = 1, \qquad i \in \{1, \ldots, r\}, \tag{3}$$

and (2) for fixed $n, r \in \mathbb{N}$, there exist positive constants $C_n(1), \ldots, C_n(r)$ such that, for each $i \in \{1, \ldots, r\}$,

$$\lim_{x\to\infty} \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} = C_n(i). \tag{4}$$

We prove in Theorem 1 that conditions (3) are sufficient for the pointwise convergence $L_n(e_i, x) \to x^i$ ($i \in \{1, \ldots, j\}$). But for studying convergence in weighted norm we need estimates of the rate of convergence in (3). We will assume that one of conditions given below holds.

For $r > 0$ set

$$\lceil r \rceil = \{\min k \in \mathbb{N} : r \leq k\}.$$

Definition 1. *For $r > 0$, we say that the sequence $\{L_n\}$ satisfies the (I, r) condition, if there exist constants $K_1, \ldots, K_{\lceil r \rceil}$ such that, for every $x > 0$, $n \in \mathbb{N}$ and $i \in \{1, \ldots, \lceil r \rceil\}$, one has*

$$\Big| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - 1 \Big| \leq \frac{K_i}{\beta(n)}. \tag{5}$$

Definition 2. *For $r > 0$, we say that the sequence $\{L_n\}$ satisfies that the (II, r) condition, if there exist constants $K_1, \ldots, K_{\lceil r \rceil}$ such that, for every $x > 0$, $n \in \mathbb{N}$ and $i \in \{1, \ldots, \lceil r \rceil\}$, one has*

$$\left| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - 1 \right| \leq \frac{K_i}{1 + \beta(n)x}. \tag{6}$$

Notice that conditions (6) implies (3) and (4).

By a weight we mean a bounded continuous function $\varrho : I \to I$. Let $C_\varrho(I)$ the family of all $f \in C(I)$ such that the function $(\varrho f)(x)$ is bounded in I. In $C_\varrho(I)$ a norm is defined by

$$\|f\|_\varrho = \sup_{x \in I} |\ (\varrho f)(x)\ | \ . \tag{7}$$

The family of all functions $f \in C_\varrho(I)$ such that $\varrho(x)f(x)$ has a finite limit as $x \to \infty$ is denoted by $C_{\varrho,\infty}(I)$.

Suppose that, for each $n \in \mathbb{N}$, $M_n : C_{\varrho,\infty}(I) \to C_{\varrho,\infty}(I)$ is a positive linear operator. We say that $\{M_n\}$ is an approximation process if, for each $f \in C_{\varrho,\infty}(I)$,

$$\lim_{n \to \infty} \|M_n(f) - f\|_\varrho = 0.$$

Throughout this work, for $r \in \mathbb{R}$, $r \geq 2$, we consider the weight

$$\varrho_r(x) = \frac{1}{(1+x)^r}. \tag{8}$$

To simplify notations we set

$$\|f\|_r = \|f\|_{\varrho_r}, \quad C_r(I) = C_{\varrho_r}(I), \quad \text{and} \quad C_{r,\infty}(I) = C_{\varrho_r,\infty}(I).$$

In this work we present sufficient conditions to verify that $\{L_n\}$ is an approximation process in $C_{r,\infty}(I)$, for $r \geq 2$. We remark that in [11] Mortici, for $f \in C_{2,\infty}(I)$, only proved convergence on compact subsets of I.

Some related results appeared in published papers for particular operators, but with some restrictions. For the weight $\varrho(x) = 1/(1+x^2)$ and a sequence a positive linear operator $M_n : C_\varrho(I) \to C_\varrho(I)$, Gadzhiev [8] proved that $\|M_n(f) - f\|_\varrho \to 0$ as $n \to \infty$, for any $C_{\varrho,\infty}(I)$. But we have not found a similar results for $r > 2$.

Ispir and Atakut in [9] studied Szász-Mirakyan operators S_n with the weight $\varrho(x) = 1/(1+x^2)$ and $S_n : C_{\varrho,\infty}(I) \to C_{\varrho^3,\infty}(I)$ (see also [6]). Notice that different weighted spaces are involved.

In Section 2 we present several properties of the operators (2) related with the test functions e_i. In Section 3 we provide sufficient conditions to verify that, for each $n \in \mathbb{N}$, $L_n : C_{r,\infty}(I) \to C_{r,\infty}(I)$. Moreover, we prove that $L_n : C_{r,\infty}(I) \to C_{r,\infty}(I)$ is a uniformly bounded sequence of operators. In Section 4 we proved that $lim_{n\to\infty} \|L_n(f_i) - f_i\|_r = 0$ for some particular functions f_i. In Section 5 that, under certain conditions, the sequence $\{L_n\}$ is an approximation process in $C_{r,\infty}(I)$. For obtaining this property the result of Section 4 are very important. Finally, in Section 6 we present several families of operators for which our approach can be applied (some of them are new).

2. Test functions and polynomials. In Proposition 1 we show that $\mathcal{D}(L)$ contains every algebraic polynomial.

Proposition 1. *Assume L_n is given by* (2).
(i) *If $f \in \mathcal{D}(L)$, then $L_n(f,0) = f(0)$.*
(ii) *Set $P_1(x) = x$ and, for $j \in \mathbb{N}$ and $x > 0$, and*

$$P_{j+1}(x) = x\Big(x - \frac{1}{\beta(n)}\Big) \cdots \Big(x - \frac{j}{\beta(n)}\Big). \tag{9}$$

For each $j \in \mathbb{N}$ one has

$$L_n(P_{j+1}, x) = x^{j+1} \frac{g_n^{(j+1)}(x)}{\beta^{j+1}(n) g_n(x)}.$$

In particular, for each $j \in \mathbb{N}_0$, $\mathbb{P}_j \subset \mathcal{D}(L)$. Moreover

$$L_n(e_1, x) = x \frac{g_n'(x)}{\beta(n) g_n(x)} \qquad \text{and} \qquad L_n(e_2, x) = x^2 \frac{g_n^{(2)}(x)}{\beta^2(n) g_n(x)} + x \frac{g_n'(x)}{\beta^2(n) g_n(x)}.$$

Proof. (i) Since $g_n(0) = a_{n,0}$, $L_n(f,0) = f(0)$, for each $f \in \mathcal{D}(L)$.
(ii) It is clear that $L_n(e_0, x) = 1$
Notice that $\beta(n) P_1(k/\beta(n)) = k$ and, for $j \in \mathbb{N}$,

$$\beta^{j+1}(n) P_{j+1}\Big(\frac{k}{\beta(n)}\Big) = k(k-1)\cdots(k-j).$$

Therefore, for each fixed $x > 0$,

$$\beta^{j+1}(n) g_n(x) L_n(P_{j+1}, x) = x^{j+1} \sum_{k=0}^{\infty} \frac{a_{n,k+j+1}}{k!} x^k = x^{j+1} g_n^{(j+1)}(x) < \infty,$$

because $g_n^{(j+1)}(x)$ is the $(j+1)$ derivative of an analytic function. Moreover

$$\beta(n)g_n(x)L_n(P_1,x) = \beta(n)g_n(x)L_n(e_1,x) = xg_n'(x),$$

and this yields the representation of $L_n(e_1,x)$. On the other hand, since $P_2(x) = x^2 - x/\beta(n)$,

$$L_n(e_2,x) = L_n(P_2,x) + \frac{1}{\beta(n)}L_n(e_1,x) = x^2\frac{g_n^{(2)}(x)}{\beta^2(n)g_n(x)} + x\frac{g_n'(x)}{\beta^2(n)g_n(x)}.$$

Since L_n is a linear operator, $\mathcal{D}(L)$ is a linear space. Hence, for each $j \in \mathbb{N}_0$, $\mathbb{P}_j \subset \mathcal{D}(L)$. □

In Proposition 1 we gave an expression for $L_n(e_1,x)$ and $L_n(e_2,x)$. In Proposition 2 we present a representation for other test functions. The proof can be obtained by induction in such a way that the exact values of the coefficients $C(i,k)$ and $D(i,k)$ are given. In this paper, we don't need to know the exact values.

Proposition 2. (i) *For each $i \geq 2$ and P_i is given by* (9), *there exist coefficients $C(i,k)$, $1 \leq k \leq i-1$, such that*

$$P_i(x) = x^i + \sum_{k=1}^{i-1}\frac{C(i,k)x^k}{\beta^{i-k}(n)}. \tag{10}$$

(ii) *For each $i \geq 2$, there exist coefficients $D(i,k)$, $1 \leq k \leq i-1$, such that*

$$L_n(e_i,x) = x^i\frac{g_n^{(i)}(x)}{\beta^i(n)g_n(x)} - H_{n,i}(x), \tag{11}$$

where

$$H_{n,i}(x) = \frac{1}{g_n(x)}\frac{1}{\beta^i(n)}\sum_{k=1}^{i-1}D(i,k)g_n^{(k)}(x)\,x^k. \tag{12}$$

Lemma 1. *Fix $n, r \in \mathbb{N}$, $r \geq 2$.*

(i) *Assume that conditions* (4) *hold. For each $i \in \{2,\dots,r\}$, there exists $N \in \mathbb{N}$ and a constant $T(i)$, depending only on i, such that*

$$\mid H_{n,i}(x) \mid \leq T(i)\frac{x^{i-1}}{\beta(n)}$$

for $x > N$.

(ii) *Assume that conditions* (3) *hold. For each* $i \in \{2, \dots, r\}$ *and* $x > 0$ *fixed*

$$\lim_{n\to\infty} H_{n,i}(x) = 0.$$

Proof. (i) It follows from conditions (4), that there exists $N \in \mathbb{N}$ such that, for every $x > N$ and $i \in \{1, \dots, r\}$,

$$\Big| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - C_n(i) \Big| \le 1.$$

Hence, for $x > N$,

$$\Big| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} \Big| = \Big| \Big(\frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - C_n(i) \Big) + C_n(i) \Big| \le 1 + C_n(i).$$

Therefore, for $x > N \ge 1$,

$$\mid H_{n,i}(x) \mid \le x^{i-1} \sum_{k=1}^{i-1} \frac{\mid D(i,k) \mid}{\beta^{i-k}(n)} \Big| \frac{g_n^{(k)}(x)}{\beta^k(n) g_n(x)} \Big| \le T(i) \frac{x^{i-1}}{\beta(n)},$$

where $T(i) = \sum_{k=1}^{i-1} \mid D(i,k) \mid (1 + C_n(k))$.

(ii) If $2 \le i \le r$,

$$\lim_{n\to\infty} H_{n,i}(x) = \sum_{k=1}^{i-1} D(i,k) x^k \Big(\lim_{n\to\infty} \frac{1}{\beta(n)^{i-k}} \frac{g_n^{(k)}(x)}{\beta^k(n) g_n(x)} \Big) = 0. \qquad \square$$

Theorem 1 (pointwise convergence)**.** *If* $r \in \mathbb{N}$, $r \ge 2$, *conditions* (3) *hold, and* $x \in I$, *then*

$$\lim_{n\to\infty} L_n(e_i, x) = x^i, \qquad i \in \{1, \dots, r\}.$$

Proof. It follows from (11) and Lemma 1. □

We have proved that, for $x \in I$ fixed, $L_n(e_1, x) \to x$ and $L_n(e_2, x) \to x^2$, as $n \to \infty$. Let us estimate some rate of convergence.

Proposition 3. *If $r \geq 2$ and $\{L_n\}$ satisfies the (I, r) condition, there exists a constant K such that, for each $n \in \mathbb{N}$ and $x \geq 0$,*

$$| L_n(e_1, x) - x | \leq \frac{Kx}{\beta(n)} \qquad \text{and} \qquad L_n((e_1 - xe_0)^2, x) \leq C\frac{x^2 + x}{\beta(n)}.$$

If $r \geq 2$ and $\{L_n\}$ satisfies the (II, r) condition, there exists a constant K such that, for each $n \in \mathbb{N}$ and $x \geq 0$,

$$| L_n(e_1, x) - x | \leq \frac{K}{\beta(n)} \qquad \text{and} \qquad L_n((e_1 - xe_0)^2, x) \leq K\frac{x}{\beta(n)}.$$

Proof. (i) From Proposition 1 and (5) we know that

$$| L_n(e_1, x) - x | = x\Big|\frac{g_n'(x)}{\beta(n)g_n(x)} - 1\Big| \leq \frac{K_1 x}{\beta(n)}$$

and

$$\begin{aligned}
&L_n((e_1 - xe_0)^2, x) = L_n(e_2, x) - 2xL_n(e_1, x) + x^2 \\
&= x^2\frac{g_n''(x)}{\beta^2(n)g_n(x)} + \frac{x}{\beta(n)}\frac{g_n'(x)}{\beta(n)g_n(x)} - 2x^2\frac{g_n'(x)}{\beta(n)g_n(x)} + x^2 \\
&= x^2\Big(\frac{g_n''(x)}{\beta^2(n)g_n(x)} - 1\Big) + \frac{x}{\beta(n)}\Big(\frac{g_n'(x)}{\beta(n)g_n(x)} - 1\Big) + \frac{x}{\beta(n)} - 2x^2\Big(\frac{g_n'(x)}{\beta(n)g_n(x)} - 1\Big) \\
&\leq C\frac{x^2 + x}{\beta(n)}.
\end{aligned}$$

(ii) From Proposition 1 and (6) we know that

$$| L_n(e_1, x) - x | = x\Big|\frac{g_n'(x)}{\beta(n)g_n(x)} - 1\Big| \leq \frac{K_1 x}{1 + \beta(n)x} \leq \frac{K_1}{\beta(n)}$$

and

$$\begin{aligned}
&L_n((e_1 - xe_0)^2, x) = L_n(e_2, x) - 2xL_n(e_1, x) + x^2 \\
&\leq \frac{K_2 x^2}{1 + \beta(n)x} + \frac{x}{\beta(n)}\frac{K_1}{(1 + \beta(n)x)} + \frac{x}{\beta(n)} + \frac{2K_1 x^2}{(1 + \beta(n)x)} \leq C\frac{x}{\beta(n)}.
\end{aligned}$$

□

3. L_n as a bounded endomorphism. First we want to find sufficient conditions to verify that, for each $n \in \mathbb{N}$,

$$L_n : C_{r,\infty}(I) \to C_{r,\infty}(I).$$

Lemma 2. *Assume $n \in \mathbb{N}$, $r > 1$, and conditions (4) are satisfied.*

(i) *If r is an integer and $1 \le i \le r-1$ or r is not an integer and $i < r$, then*

$$\lim_{x\to\infty} \varrho_r(x) L_n(e_i, x) = 0.$$

(ii) *If r is an integer,*

$$\lim_{x\to\infty} \varrho_r(x) L_n(e_r, x) = C_n(r) \qquad \text{and} \qquad \lim_{x\to\infty} \varrho_r(x) L_n(1/\varrho_r, x) = C_n(r).$$

Proof. (i) Taking into account (4), we fix $N_1 \in \mathbb{N}$ such that, for $x > N_1$ and $1 \le i \le r-1$,

$$\Big| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - C_n(i) \Big| \le 1.$$

We first consider the case $i = 1$. Since

$$0 \le \varrho_r(x) L_n(e_1, x) \le \varrho_r(x) \mid L_n(e_1, x) - C_n(1)x \mid + \mid C_n(1) \mid \varrho_r(x) x$$

$$= \frac{x}{(1+x)^r} \Big\{ \Big| \frac{g_n'(x)}{\beta(n) g_n(x)} - C_n(1) \Big| + \mid C_n(1) \mid \Big\}.$$

and it is sufficient to prove the result.

If r is an integer and $2 \le i \le r-1$, we take $N > N_1$ as in Lemma 1. It follows from equation (11) that

$$0 \le \varrho_r(x) L_n(e_i, x) \le \varrho_r(x) \mid L_n(e_i, x) - C_n(i) x^i \mid + \mid C_n(i) \mid \varrho_r(x) x^i$$

$$\le \frac{x^i}{(1+x)^r} \Big| \frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - C_n(i) \Big| + \varrho_r(x) \mid H_{n,i}(x) \mid + \mid C_n(i) \mid \frac{x^i}{(1+x)^r}.$$

$$\le \frac{x^i}{(1+x)^r} + \varrho_r(x) T(i) \frac{x^{i-1}}{\beta(n)} + \mid C_n(i) \mid \frac{x^i}{(1+x)^r} \le C \frac{x^i}{(1+x)^r}.$$

If r is not an integer the proof follows analogously.

(ii) If r is an integer, from Lemma 1 we know that, for $x > N$,

$$\varrho_r(x) \mid H_{n,r}(x) \mid \le T(r) \frac{1}{(1+x)}.$$

Hence $\lim_{x\to\infty} \varrho_r(x) \mid H_{n,r}(x) \mid = 0$ and it follows from conditions (4) that

$$\lim_{x\to\infty} \varrho_r(x)x^r \Big| \frac{g_n^{(r)}(x)}{\beta^r(n)g_n(x)} - C_n(r)\Big| = \lim_{x\to\infty} \Big| \frac{g_n^{(r)}(x)}{\beta^r(n)g_n(x)} - C_n(r)\Big| = 0.$$

Since, taking into account (11), we have

$$\varrho_r(x)\Big(L_n(e_r, x) - x^r C_n(r)\Big) = \varrho_r(x)\Big(x^r\Big(\frac{g_n^{(r)}(x)}{\beta^r(n)g_n(x)} - C_n(r)\Big) - H_{n,r}(x)\Big),$$

we have proved that $\lim_{x\to\infty} \varrho_r(x)L_n(e_r, x) = C_n(r)$.

Finally

$$\lim_{n\to\infty} \varrho_r(x)\sum_{j=0}^{r}\binom{r}{j}L_n(e_j, x) = \lim_{n\to\infty} \varrho_r(x)L_n(e_r, x) = C_n(r). \qquad \square$$

Theorem 2. *Assume $r > 1$. If conditions* (4) *are satisfied and $f \in C_{r,\infty}(I)$, then $L_n(f) \in C_{r,\infty}(I)$, for every $n \in \mathbb{N}$.*

P r o o f. Fix $n \in \mathbb{N}$. If $f \in C_{r,\infty}[0, \infty)$ there exists a real A such that $\varrho_r(x)f(x) \to A$, as $x \to \infty$. We set $B = 1 + \mid A \mid + \|f\|_r$.

Taking into account Lemma 2, there exists $N_1 > 0$ such that, for $x > N_1$,

$$\varrho_r(x)L_n(1/\varrho_r, x) \le 2C_n(r).$$

Given $\varepsilon > 0$, there exists $N_2 > N_1$ such that, for $x > N_2$,

$$\mid \varrho_r(x)f(x) - A \mid < \frac{\varepsilon}{2(1 + 2C_n(r))}.$$

Since $k/\beta(n) \to \infty$ as $k \to \infty$, there exists $q \in \mathbb{N}$ such that $k/\beta(n) > N_2$, for all $k > q$.

Taking into account L'Hôpital's rule

$$\lim_{x\to\infty} \frac{1}{g_n(x)} \sum_{k=0}^{q} \frac{a_{n,k}}{k!} \frac{x^k}{\varrho_r(k/\beta(n))} = 0.$$

Hence, there exists $N_3 > N_2$ such that, for $x > N_3$,

$$\frac{1}{g_n(x)} \sum_{k=0}^{q} \frac{a_{n,k}}{k!} \frac{x^k}{\varrho_r(k/\beta(n))} \le \frac{\varepsilon}{2B}.$$

Therefore, if $x > N_3$, then

$$\begin{aligned}
&\mid \varrho_r(x)L_n(f,x) - A\varrho_r(x)L_n(1/\varrho_r,x) \mid \\
&\quad = \left| \frac{\varrho_r(x)}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} \frac{(\varrho_r(k/\beta(n))f(k/\beta(n)) - A)}{\varrho_r(k/\beta(n))} x^k \right| \\
&\quad \leq \frac{\varrho_r(x)}{g_n(x)} \sum_{k=0}^{q} \left| \frac{a_{n,k}}{k!} \frac{(\varrho_r(k/\beta(n))f(k/\beta(n)) - A)}{\varrho_r(k/\beta(n))} x^k \right| \\
&\qquad + \left| \frac{\varrho_r(x)}{g_n(x)} \sum_{k=q+1}^{\infty} \frac{a_{n,k}}{k!} \frac{(\varrho_r(k/\beta(n))f(k/\beta(n)) - A)}{\varrho_r(k/\beta(n))} x^k \right| \\
&\quad \leq (\|f\|_r + \mid A \mid) \frac{\varrho_r(x)}{g_n(x)} \sum_{k=0}^{q} \frac{a_{n,k}}{k!} \frac{x^k}{\varrho_r(k/\beta(n))} \\
&\qquad + \frac{\varepsilon}{2(1+2C_m)} \frac{\varrho_r(x)}{g_n(x)} \sum_{k=q+1}^{\infty} \frac{a_{n,k}}{k!} \frac{x^k}{\varrho(k/\beta(n))} \\
&\quad \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2(1+2C_n(r))} \varrho_r(x)L_n(1/\varrho_r,x) < \varepsilon.
\end{aligned}$$

This proves that

$$\begin{aligned}
\lim_{x\to\infty} \varrho_r(x)L_n(f,x) &= \lim_{x\to\infty} \varrho_r(x)\Big(L_n(f,x) - AL_n(1/\varrho_r,x) + AL_n(1/\varrho_r,x)\Big) \\
&= A \lim_{x\to\infty} \varrho_r(x)L_n(1/\varrho_r,x) = AC_n(r).
\end{aligned}$$

Finally, since the series $L_n(f,x)$ converges absolutely and uniformly on each compact subset of I, then $L_n(f) \in C(I)$. □

Theorem 3. *Assume $r > 0$, and $\{L_n\}$ satisfies the (I,r) (or (II,r)) condition.*

(i) *One has $(1+e_1)^r \in \mathcal{D}(L)$ and there exists a positive constant Λ_r (which depends only on r) such that, for each $n \in \mathbb{N}$ and $x \in I$,*

$$L_n((1+e_1)^r, x) \leq \Lambda_r(1+x)^r.$$

(ii) *If $f \in C_r(I)$, then $L_n(f) \in \mathcal{D}(L)$ and*

$$\|L_n(f)\|_r \leq \Lambda_r\|f\|_r.$$

Proof. (i) First assume that r is an integer. If (5) holds, K_i are the constants given in (5) and $1 < i \le r$, it follows from (11) that

$$0 \le L_n(e_i, x) = x^i \Big(\frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - 1 \Big) + x^i,$$

$$- \sum_{k=1}^{i-1} D(i,k) \frac{1}{\beta^{i-k}(n)} \Big(\frac{g_n^{(k)}(x)}{\beta^k(n) g_n(x)} - 1 \Big) x^k - \sum_{k=1}^{i-1} D(i,k) \frac{1}{\beta^{i-k}(n)}\, x^k$$

$$\le (1 + K_i) x^i + \Big(1 + \max_{1 \le k \le i-1} K(k) \Big) \sum_{k=1}^{i-1} \mid D(i,k) \mid \frac{1}{\beta^{i-k}(n)}\, x^k.$$

On the other hand $L_n(e_0, x) = 1$ and

$$L_n(e_1, x) = x \Big(\frac{g_n'(x)}{\beta(n) g_n(x)} - 1 \Big) + x \le Cx.$$

In particular, we have proved that $L_n(e_i) \in C(I)$.

If we assume (6), the inequality above can be written as

$$L_n(e_i, x) \le x^i + K_i\, x^{i-1} + \Big(1 + \max_{1 \le k \le i-1} K(k) \Big) \sum_{k=1}^{i-1} \mid D(i,k) \mid \frac{1}{\beta^{i-k}}\, x^{k-1}.$$

On the other hand,

$$L_n(e_1, x) = x \Big(\frac{g_n'(x)}{\beta(n) g_n(x)} - 1 \Big) + x \le \frac{K_1}{\beta(n)} + x.$$

In both cases, if $x < 1$, there exists a constant $C(i)$, depending only on i, such that $0 \le L_n(e_i, x) \le C(i)$ and, if $x \ge 1$, $0 \le L_n(e_i, x) \le C(i) x^i$.

Therefore, if $r \in \mathbb{N}$ and $x < 1$

$$L_n((1 + e_1)^r, x) = \sum_{i=0}^{r} \binom{r}{i} L_n(e_i, x) \le C_1 \sum_{i=0}^{r} \binom{r}{i} \le C_2 (1 + x)^r$$

and, if $x \ge 1$, then

$$L_n((1 + e_1)^r, x) \le C \sum_{i=0}^{r} \binom{r}{i} x^i = C(1 + x)^r.$$

If $r > 0$ is not an integer, then

$$L_n((1+e_1)^r, x) \leq \Big(L_n((1+e_1)^{\lceil r\rceil}, x)\Big)^{r/\lceil r\rceil} \leq C(1+x)^r.$$

(ii) If $f \in C_r(I)$ and $x \in I$, then

$$\varrho_r(x) \mid L_n(f,x) \mid \leq \|f\|_r \varrho_r(x) L_n((1+e_1)^r, x) \leq \Lambda_r \|f\|_r. \qquad \square$$

4. Approximation of several particular functions. Set

$$g_{r,i}(x) = (1+x)^{r-i}, \quad x \in I, \quad 0 \leq i \leq 2.$$

We need to verify that, for $i \in \{0,1,2\}$,

$$\lim_{n\to\infty} \|L_n(g_{r,i}) - g_{r,i}\|_r = 0.$$

We will use the representation (derived from Taylor's formula)

$$L_n(g_{r,i}, x) - g_{r,i}(x) = (r-i) L_n\Big(\int_x^{e_1} (1+s)^{r-i-1} ds, x\Big). \tag{13}$$

Since different arguments will be used we separate the following cases:
A) $i = 0$.
B) $i \in \{1,2\}$ and $r - i - 1 = 0$.
C) $i \in \{1,2\}$ and $r - i - 1 > 0$
D) $2 \leq r < 3$ and $i = 2$.

Notice since $r - 2 \geq 0$, for $i = 1$ the cases B) and C) cover all possible relations.

The case $i = 2$ and $r = 3$ is included in B). The case $i = 2$ and $r > 3$ is included in C) and D) covers the remaining cases.

Notice that if $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition and $m < 2r$, then $\{L_n\}$ satisfies the (I, m) (or (II, m)) condition.

To avoid repetitions we first prove the following lemma.

Lemma 3. *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $i \in \{0,1,2\}$ and $r - i - 1 \geq 0$, there exists a constant C such that, if $\beta(n)x < 1$, then*

$$\varrho_r(x) \mid L_n(g_{r,i}, x) - g_{r,i}(x) \mid \leq \frac{C}{\sqrt{\beta(n)}}. \tag{14}$$

P r o o f. Notice that

$$\sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} \Big| \int_x^{k/\beta(n)} g'_{r,i}(s) ds \Big| x^k = (r-i) \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} \Big| \int_x^{k/\beta(n)} (1+s)^{r-i-1} ds \Big| x^k.$$

If $\beta(n)x < 1$ and $k = 0$ one has

$$r a_{n,0} \int_0^x (1+s)^{r-i-1} ds \le r a_{n,0} (1+x)^{r-i-1} x \le r g_n(x) \frac{(1+x)^{r-i-1}}{\beta(n)}.$$

Moreover, taking into account Theorem 3 (with $2(r-i-1)$ instead of r) and Hölder inequality we obtain

$$\begin{aligned} &\frac{1}{g_n(x)} \sum_{k=1}^{\infty} \frac{a_{n,k} x^k}{k!} \int_x^{k/\beta(n)} (1+s)^{r-i-1} ds \\ &\quad \le \frac{1}{g_n(x)} \sum_{k=1}^{\infty} \frac{a_{n,k} x^k}{k!} (1 + k/\beta(n)^{r-i-1} (k/\beta(n) - x) \\ &\quad \le \Big(L_n((1+e_1)^{2(r-i-1)}, x) \Big)^{1/2} \Big(L_n((e_1 - x e_0)^2, x) \Big)^{1/2} \\ &\quad \le C (1+x)^{r-i-1} \Big(L_n((e_1 - x e_0)^2, x) \Big)^{1/2}. \end{aligned}$$

If (5) holds (recall that $\beta(n)x < 1$), it follows from Proposition 3 that

$$\frac{1}{g_n(x)} \sum_{k=1}^{\infty} \frac{a_{n,k} x^k}{k!} \int_x^{k/\beta(n)} (1+s)^{r-i-1} ds \le C_1 \frac{(1+x)^{r-i-1}}{\sqrt{\beta(n)}}.$$

Therefore

$$\varrho_r(x) \mid L_n(g_{r,i}, x) - g_{r,i}(x) \mid \le \frac{C}{\sqrt{\beta(n)}}. \tag{15}$$

If (6) holds, it follows from Proposition 3 that

$$\frac{1}{g_n(x)} \sum_{k=1}^{\infty} \frac{a_{n,k} x^k}{k!} \int_x^{k/\beta(n)} (1+s)^{r-i-1} ds \le C_1 (1+x)^{r-i-1} \frac{1}{\sqrt{\beta(n)}}.$$

and we obtain again (15). □

Lemma 4 (Case A). *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. There exists a constant C such that, if $n \in \mathbb{N}$ and $x \in I$, then (14) holds with $i = 0$.*

Proof. If $i = 0$, $r - i - 1 = r - 1 > 0$ and we can apply Lemma 3 when $\beta(n)x \leq 1$. In what follows we assume that $\beta(n)x > 1$

Notice that $g'_{r,0}(x) = r(1+x)^{r-1}$. Hence

$$\frac{1}{g_n(x)}\frac{1}{r}\sum_{k=0}^{\infty}\frac{a_{n,k}x^k}{k!}\Big|\int_x^{k/\beta(n)} g'_{r,0}(s)ds\Big|$$

$$\leq \frac{1}{g_n(x)}\Big\{\sum_{k=0}^{\infty}\frac{a_{n,k}x^k}{k!}\Big|\frac{k}{\beta(n)} - x\Big|\Big((1+x)^{r-1} + \Big(1+\frac{k}{\beta(n)}\Big)^{r-1}\Big)\Big\}$$

$$\leq \sqrt{L_n((e_1 - xe_0)^2, x)}\Big((1+x)^{r-1} + (L_n((1+e_1)^{2(r-1)}, x))^{1/2}\Big)$$

$$\leq C(1+x)^{r-1}\sqrt{L_n((e_1 - xe_0)^2, x)}.$$

Using Proposition 3 we obtain the desired inequality. □

Lemma 5 (Case B). *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $i \in \{1, 2\}$ and $r - i - 1 = 0$, there exists a constant C such that, if $n \in \mathbb{N}$ and $x \in I$, then (14) with $i = 1$ and $i = 2$.*

Proof. If $i = 1$, $r = 2$ and $g_{2,1}(x) = 1 + x$. In such a case $L_n(g_{2,1}, x) - g_{2,1}(x) = L_n(e_1, x) - x$ and it follows from Proposition 3 that

$$\varrho_2(x) \mid L_n(g_{2,1}, x) - g_{2,1}(x) \mid \leq \frac{1}{(1+x)^2}\frac{Kx}{\beta(n)} \leq \frac{K}{\sqrt{\beta(n)}}.$$

If $i = 2$, then $r = 3$ and $g_{3,2}(x) = (1 + x)$. Hence $L_n(g_{3,2}, x) - g_{3,2}(x) = L_n(e_1, x) - x$ and the proof follows as in the case $i = 1$. □

Proposition 4. *Fix $r > 1$. For $x, t > 0$,*

$$\Big|\int_x^t \frac{du}{u^{1/2}\varrho_r(u)}\Big| \leq 2\frac{\mid t - x \mid}{\sqrt{x}}\Big((1+x)^r + (1+t)^r\Big). \tag{16}$$

Proof. If $t < x$,

$$\int_t^x \frac{(1+u)^r}{u^{1/2}}\,du \leq (1+x)^r\int_t^x \frac{1}{u^{1/2}}\,du = 2(\sqrt{x} - \sqrt{t})(1+x)^r \leq 2\frac{x-t}{\sqrt{x}}(1+x)^r.$$

On the other hand, if $x < t$,

$$\int_x^t \frac{(1+u)^r}{u^{1/2}}\,du \le \frac{(1+t)^r}{\sqrt{x}}(t-x). \qquad \square$$

Lemma 6 (Case C). *Assume $r \in \mathbb{R}$, $r \ge 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $i \in \{1, 2\}$ and $r - i - 1 > 0$, there exists a constant C such that, if $n \in \mathbb{N}$ and $x \in I$, then* (14) *with $i = 1$ and $i = 2$.*

P r o o f. As before, we should consider that $\beta(n)x > 1$.

Notice that, for $i \in \{1, 2\}$,

$$\sqrt{x}g'_{r,i}(x) = (r-i)\sqrt{x}(1+x)^{r-i-1} \le (r-i)g_{r,i}(x).$$

Hence $g'_{r,i}(x) \le (r-i)g_{r,i}(x)/\sqrt{x}$.

In this case we use (16) (with $r - i$ in place of r) and Theorem 3 to obtain

$$\begin{aligned}
&\frac{\varrho_r(x)}{g_n(x)}\sum_{k=0}^{\infty}\frac{a_{n,k}x^k}{k!}\Big|\int_x^{k/\beta(n)} g'_{r,i}(s)ds\Big| \\
&\quad\le \frac{r\varrho_r(x)}{g_n(x)}\sum_{k=0}^{\infty}\frac{a_{n,k}x^k}{k!}\Big|\int_x^{k/\beta(n)} \frac{(1+s)^{r-i}ds}{\sqrt{s}}\Big| \\
&\le \frac{2r\varrho_r(x)}{g_n(x)\sqrt{x}}\sum_{k=0}^{\infty}\frac{a_{n,k}x^k}{k!}\Big(\Big|\frac{k}{\beta(n)}-x\Big|\Big((1+x)^{r-i}+\Big(1+\frac{k}{\beta(n)}\Big)^{r-i}\Big)\Big) \\
&\le \frac{2r\varrho_r(x)}{\sqrt{x}}\sqrt{L_n((e_1-xe_0)^2,x)}\Big((1+x)^{r-i}+\sqrt{L_n((1+e_1)^{2(r-i)},x)}\Big) \\
&\quad\le C\frac{1}{\sqrt{x}(1+x)^i}\sqrt{L_n((e_1-xe_0)^2,x)},
\end{aligned}$$

and the result follows from Proposition 3. $\square$

Lemma 7 (Case D). *Assume $r \in \mathbb{R}$, $r \ge 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $2 \le r < 3$ and $i = 2$, there exists a constant C such that, if $n \in \mathbb{N}$ and $x \in I$, then*

$$\varrho_r(x) \mid L_n(g_{r,2},x) - g_{r,2}(x) \mid = 0, \qquad \textit{if} \quad r = 2,$$

and

$$\varrho_r(x) \mid L_n(g_{r,2},x) - g_{r,2}(x) \mid \le \frac{C}{\beta(n)^{(r-2)/2}}, \qquad \textit{if} \quad r > 2.$$

Proof. If $r = 2$ the assertion is trivial, because $g_{r,2}(x) = (1+x)^{r-2} = 1$.

If $2 < r < 3$, then $0 < r-2 < 1$. Recall that, if $a, b \geq 0$ and $0 < \alpha < 1$, then $| a^\alpha - b^\alpha | \leq | a - b |^\alpha$. Hence

$$\left| L_n((1+e_1)^{r-2}, x) - (1+x)^{r-2} \right| \leq \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} \left| \Big(1 + \frac{k}{\beta(n)}\Big)^{r-2} - (1+x)^{r-2} \right| x^k$$

$$\leq \frac{1}{g_n(x)} \sum_{k=0}^{\infty} \frac{a_{n,k}}{k!} \left| \frac{k}{\beta(n)} - x \right|^{r-2} x^k \leq \Big(L_n\Big(\mid e_1 - xe_0 \mid, x\Big)\Big)^{r-2}$$

$$\leq \Big(L_n\Big((e_1 - xe_0)^2, x\Big)\Big)^{(r-2)/2}.$$

Therefore, if conditions (5) hold, then

$$\varrho_r(x) \left| L_n(g_{r,2}, x) - g_{r,2}(x) \right| \leq \frac{C}{(1+x)^r} \frac{x^{r-2} + x^{(r-2)/2}}{\beta(n)^{(r-2)/2}} \leq \frac{C_1}{\beta(n)^{(r-2)/2}}.$$

A similar inequality is obtained if conditions (6) hold. □

From the previous Lemmas we obtain a first important result.

Proposition 5. *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $i \in \{0, 1, 2\}$, then*

$$\lim_{n\to\infty} \| L_n((1+e_1)^{r-i}, x) - (1+x)^{r-i} \|_r = 0.$$

Now we prove the main result of this section.

Theorem 4. *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If $f_i(x) = x^i(1+x)^{r-i}$, for $i \in \{0, 1, 2\}$, then*

$$\lim_{n\to\infty} \| L_n(f_i) - f_i \|_r = 0.$$

Proof. The case $i = 0$ is included in Proposition 5.

For $i = 1$, we use the identity $x(1+x)^{r-1} = (1+x)^r - (1+x)^{r-1}$, to obtain

$$L_n(e_1(1+e_1)^{r-1}, x) - x(1+x)^{r-1}$$
$$= L_n((1+e_1)^r, x) - (1+x)^r - \Big(L_n((1+e_1)^{r-1}, x) - (1+x)^{r-1} \Big).$$

Hence, the assertion follows again from Proposition 5.

For $i = 2$, the proof follows analogously, because

$$x^2(1+x)^{r-2} = (1+x)^r - 2(1+x)^{r-1} + (1+x)^{r-2}.$$ □

5. Approximation process. Now we will study convergence in the spaces $C_{r,\infty}(I)$. We consider the space $C_{r,\infty}(I)$ instead of $C_r(I)$ because we use an isomorphism between $C_{r,\infty}(I)$ and $C[0,1]$. Some authors forget that, for any weight ϱ, the assumption that $\lim\limits_{x\to\infty}(\varrho f)(x)$ exists is necessary for the existence of the isomorphism (for instance, see [7, p. 152]).

We first recall a known result.

Theorem 5 (see [2] and [3]). *Let ϱ be an arbitrary weight. A sequence $\{M_n\}$ of positive linear operators, $M_n : C_{\varrho,\infty}(I) \to C_{\varrho,\infty}(I)$, is an approximation process if and only if $\|f_i - M_n(f_i)\|_\varrho \to 0$, for $i = 0, 1, 2$, where*

$$(17)\qquad f_0(x) = \frac{1}{\varrho(x)}, \quad f_1(x) = \frac{x}{\varrho(x)(1+x)}, \ \text{and}\ f_2(x) = \frac{x^2}{\varrho(x)(1+x)^2}.$$

For the next result recall that (6) implies (4).

Theorem 6. *Assume $r \in \mathbb{R}$, $r \geq 2$, and $\{L_n\}$ satisfies the $(I, 2r)$ (or $(II, 2r)$) condition. If conditions (4) are satisfied, then $\{L_n\}$ is an approximation process in $C_{r,\infty}(I)$.*

Proof. From Theorem 2 we know that $L_n : C_{r,\infty}(I) \to C_{r,\infty}(I)$.

For the weight $\varrho_r(x) = 1/(1+x)^r$, the function f_0, f_1 and f_2 in Theorem 5 can be written as

$$f_0(x) = (1+x)^r, \quad f_1(x) = x(1+x)^{r-1} \quad \text{and} \quad f_2(x) = x^2(1+x)^{r-2}.$$

It was proved in Theorem 4 that $\lim\limits_{n\to\infty} \|L_n(f_i) - f_i\|_r = 0$, for $i = 0, 1, 2$. Hence, as a consequence of Theorem 5, one has that $\{L_n\}$ is an approximation process in $C_{r,\infty}(I)$. □

Let us show how one can construct different sequence of operators satisfying the conditions of Theorem 6.

Theorem 7. *Fix $\in \mathbb{R}$, $r \geq 2$. Let $\{b_k\}$ be a decreasing sequence of positive real numbers. Suppose that, for each $i \in \mathbb{N}$, there exists a constant $C(i)$ such that, for every $k \in \mathbb{N}$, one has*

$$(18)\qquad b_{k-1} - b_{k-1+i} \leq \frac{C(i)}{k} b_k.$$

If

$$(19)\qquad B_n(f, x) = \frac{1}{q_n(x)} \sum_{k=0}^{\infty} \frac{n^k b_k}{k!} f\Big(\frac{k}{n}\Big) x^k \quad \text{where} \quad q_n(x) = \sum_{k=0}^{\infty} \frac{n^k b_k}{k!} x^k,$$

then the sequence $\{B_n\}$ is an approximation process in $C_{r,\infty}(I)$.

Proof. In order to apply Theorem 6 we will verify that the inequalities in (6) hold for every $i \in \mathbb{N}$.

Since

$$q_n'(x) = \sum_{k=1}^{\infty} \frac{n^k b_k}{(k-1)!} x^{k-1} = n \sum_{k=0}^{\infty} \frac{n^k b_{k+1}}{k!} x^k,$$

it follows by induction that

$$q_n^{(i)}(x) = n^i \sum_{k=0}^{\infty} \frac{b_{k+i}}{k!}(nx)^k = n^i \sum_{k=0}^{\infty} \frac{b_{k+i} - b_k}{k!}(nx)^k + n^i q_n(x).$$

Hence

$$\Big|\frac{q_n^{(i)}(x)}{n^i q_n(x)} - 1\Big| = \frac{1}{q_n(x)} \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!}(nx)^k,$$

because $\{b_k\}$ decreases.

Taking into account (18) we obtain

$$\begin{aligned}
&(1+nx) \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!}(nx)^k \\
&\quad = \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!}(nx)^k + \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!}(nx)^{k+1} \\
&\quad = \sum_{k=0}^{\infty} \frac{b_k - b_{k+i}}{k!}(nx)^k + \sum_{k=1}^{\infty} \frac{b_{k-1} - b_{k-1+i}}{(k-1)!}(nx)^k \\
&\quad \le \sum_{k=0}^{\infty} \frac{b_k}{k!}(nx)^k + C(i) \sum_{k=1}^{\infty} \frac{b_k}{k!}(nx)^k \le (1 + C(i)) q_n(x).
\end{aligned}$$

Therefore

$$\Big|\frac{g_n^{(i)}(x)}{n^i g_n(x)} - 1\Big| \le \frac{1 + C(i)}{1 + nx}. \qquad \square$$

There is a result similar to Theorem 7, but for an increasing sequences. In this case we should take care that the condition in the right side of (1) is satisfied. But, what we really need is a control of how fast the sequence increases.

Theorem 8. *Fix $r \in \mathbb{N}$, $r \geq 2$. Let $\{c_k\}$ be an increasing sequence of positive real numbers. Suppose that, for each $i \in \mathbb{N}$, there exists constants $C_1(i)$ and $C_2(i)$ such that, for every $k \in \mathbb{N}_0$, one has*

$$c_{k+i} - c_k \leq C_1(i)\frac{c_{k+1}}{k+1} \qquad \text{and} \qquad c_{k+i} - c_k \leq C_2(i)c_k. \tag{20}$$

If

$$C_n(f,x) = \frac{1}{r_n(x)}\sum_{k=0}^{\infty}\frac{n^k c_k}{k!}f\Big(\frac{k}{n}\Big)x^k \quad \text{where} \quad r_n(x) = \sum_{k=0}^{\infty}\frac{n^k c_k}{k!}x^k, \tag{21}$$

then the sequence $\{C_n\}$ is an approximation process in $C_{r,\infty}(I)$.

Proof. The proof follows as in the previous theorem, with $g_n(x) = r_n(x)$, but in this case

$$(1+nx)\sum_{k=0}^{\infty}\frac{c_{k+i}-c_k}{k!}(nx)^k = \sum_{k=0}^{\infty}\frac{c_{k+i}-c_k}{k!}(nx)^k + \sum_{k=0}^{\infty}\frac{c_{k+i}-c_k}{k!}(nx)^{k+1}$$

$$= \sum_{k=0}^{\infty}\frac{c_{k+i}-c_k}{k!}(nx)^k + \sum_{k=1}^{\infty}\frac{c_{k-1+i}-c_{k-1}}{(k-1)!}(nx)^k$$

$$\leq C_2(i)\sum_{k=0}^{\infty}\frac{c_k}{k!}(nx)^k + C_1(i)\sum_{k=1}^{\infty}\frac{c_k}{(k-1)!k}(nx)^k \leq (C_1(i)+C_2(i))g_n(x).$$

Therefore

$$\Big|\frac{g_n^{(i)}(x)}{n^i g_n(x)} - 1\Big| \leq \frac{C_1(i)+C_2(i)}{1+nx}. \qquad \square$$

6. Applications. In this section we present several examples of process for which the result of Section (5) can be applied. In each case we will verify the conditions in the respective theorem, but we will not formulate the corresponding theorem.

Example 1. For a fixed $p \geq 0$, Schurer introduced the operators

$$S^*_{n,p}(f,x) = e^{-(n+p)x}\sum_{k=0}^{\infty}\frac{(n+p)^k}{k!}f\Big(\frac{k}{n}\Big)x^k. \tag{22}$$

The case $p = 0$ gives play to the Szász-Mirakyan operators. The operators $S^*_{n,p}$ have been studied by some authors (see [14] and [15]). Moreover, these operators are included in a class studied [9].

In this work we consider the more general version

$$S_{n,p}(f,x) = e^{-(\beta(n)+p)x} \sum_{k=0}^{\infty} \frac{(\beta(n)+p)^k}{k!} f\Big(\frac{k}{\beta(n)}\Big) x^k, \tag{23}$$

where $\beta(n) \geq 1$ and $\beta(n) \to \infty$, as $n \to \infty$. In order to apply Theorem 6 we will verify (4) and (5) hold for every $i \in \mathbb{N}$.

Notice that

$$\frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} = \frac{(\beta(n)+p)^i}{\beta^i(n)} = \Big(1 + \frac{p}{\beta(n)}\Big)^i. \tag{24}$$

Hence conditions (4) hold for every $i \in \mathbb{N}$.

If we use the expansion

$$\Big(\frac{\beta(n)+p}{\beta(n)}\Big)^i = \Big(1 + \frac{p}{\beta(n)}\Big)^i = 1 + \sum_{j=1}^{i} \binom{i}{j} \frac{p^j}{\beta^j(n)},$$

we obtain (recall that $\beta(n) \geq 1$)

$$\Big|\frac{g_n^{(i)}(x)}{\beta^i(n) g_n(x)} - 1\Big| = \sum_{j=1}^{i} \binom{i}{j} \frac{p^j}{\beta^j(n)} \leq \frac{1}{\beta(n)} \sum_{j=1}^{i} \binom{i}{j} p^j \leq \frac{K_i}{\beta(n)}.$$

Hence conditions (5) are satisfied for every $i \in \mathbb{N}$.

We have proved that the operators $\{S_{n,p}\}$ is an approximation process in $C_{r,\infty}[0,\infty)$, with $r \geq 2$. We have not found a similar result in works devoted to study these operators . For instance, in [10] the authors consider the weighted space with $r = 2$, but nothing is said concerning convergence in norm (see also [12]).

Example 2. Let $\{\alpha(n)\}$ be a sequence such that $\alpha(n) \geq 1$ and $\alpha(n) \to \infty$, as $n \to \infty$. For $p \geq 0$, $x \geq 0$, and a function $f \in C(I)$ define

$$M_{n,p}(f,x) = e^{-(\alpha(n)+p)x} \sum_{k=0}^{\infty} \frac{(\alpha(n)+p)^k}{k!} f\Big(\frac{k}{\alpha(n)+p}\Big) x^k \tag{25}$$

whenever the series converges absolutely. The operator (22) do not reproduce affine functions, while the operators (25) have this property.

The operators $M_{n,p}$ were studied by Doğru in [6], where the author write $h(n)$ instead $\alpha(n)+p$. In the quoted work the author estimate the error of approximation with respect to the weight $1+x^2$, for functions in the weighted space $C_{\varrho,\infty}(I)$, with $\varrho(x)=(1+x^2)^{2/3}$. That is two different weighted spaces are involved.

In [4] and [5] the authors studies the operators $M_{n,p}$ in a more general approach by considering the weights (8) (without working with different weighted spaces).

Notice that

$$\frac{g_{n,p}^{(i)}(x)}{\beta^i(n)g_{n,p}(x)}=\Big(\frac{\alpha(n)+p}{\alpha(n)+p}\Big)^i=1.$$

Hence conditions (4) hold with $C_n(i)=1$ and conditions (5) are satisfied with $K_i=0$.

Let us remark that the definition of the operators (25) agrees with the method used in [1] to construct positive linear operators reproducing affine functions. In fact, in [1] the operators were defined by

$$R_n(f,x)\;=\;\frac{1}{g_n(x)}\sum_{k=0}^{\infty}\frac{a_{n,p}^k}{k!}f\left(\frac{ka_{n,p}^{k-1}}{a_{n,p}^k}\right)x^k,$$

with the convection $a_{n,p}^{-1}=0$. But, if $a_{n,p}=\alpha(n)+p$, for $k\geq 1$ one has

$$\frac{a_{n,p}^{k-1}}{a_{n,p}^k}=\frac{1}{\alpha(n)+p}.$$

Example 3. We present a first example of a family of operators which are not included in Examples 1 and 2. For fixed $r,n\in\mathbb{N}$ and $x\in I$, $f\in C(I)$ and $x\in I$, define

$$N_{n,r}(f,x)=\frac{1}{\widetilde{N}_{n,r}(x)}\sum_{k=0}^{\infty}\frac{n^k(k+1)^r}{k!}f\Big(\frac{k}{n}\Big)x^k,\quad \widetilde{N}_{n,r}(x)=\sum_{k=0}^{\infty}\frac{n^k(k+1)^r}{k!}x^k,$$

whenever the first series converges absolutely.

The operators $N_{n,2}(f,x)$ seen to be simpler that the operators

$$(\varphi S_nf)(x)=\frac{1}{(nx+1)^2e^{nx}}\sum_{k=0}^{\infty}\frac{n^k(k^2+k+1)}{k!}f\Big(\frac{k}{n}\Big)x^k,\quad x\in I,$$

considered by Pop et al in [13].

Taking into account Theorem 8, we will verify that conditions (20) hold.
Set $c_k = (k+1)^r$.
If $k = 0$, $c_{k+i} - c_k = (i+1)^r - 1 \le (i+1)^r(k+1)^r$.
If $k \ge 1$, then

$$c_{k+i} - c_k = (k+i+1)^r - (k+1)^r = \sum_{j=0}^{r} \binom{r}{j} i^{r-j}(k+1)^j - (k+1)^r$$

$$= \sum_{j=0}^{r-1} \binom{r}{j} i^{r-j}(k+1)^j \le (2i)^r \sum_{j=0}^{r-1} \binom{r}{j} k^j \le (2i)^r(1+k)^r = (2i)^r c_k.$$

On the other hand, by the mean value theorem, there exists θ, $k+1 < \theta < k+i+1$, such that

$$c_{k+i} - c_k = (k+i+1)^r - (k+1)^r = ri\theta^{r-1} \le ri(k+i+1)^{r-1}$$

$$\le ri^r(k+2)^{r-1} = ri^r \frac{(k+2)^r}{k+2} \le ri^r \frac{c_{k+1}}{k+1}.$$

Therefore the two inequalities in (20) hold.

It can be proved that there exists an algebraic polynomial $R_{r-1}(x)$ of degree non greater that $r-1$ with with positive coefficients such that

$$\widetilde{N}_{n,r}(x) = \Big(R_{r-1}(x) + x^r\Big)e^x,$$

but will not include the proof here. The study of operators satisfying the above relation were suggested by Mortici in [11].

Example 4. For $0 < \gamma < 1$, $n \in \mathbb{N}$, $f \in C(I)$, and $x \in I$, define

$$Q_{n,\gamma}(f,x) = \frac{1}{q_{n,\gamma}(x)} \sum_{k=0}^{\infty} \frac{n^k(k+1)^\gamma}{k!} f\Big(\frac{k}{n}\Big)x^k, \quad q_{n,\gamma}(x) = \sum_{k=0}^{\infty} \frac{n^k(k+1)^\gamma}{k!} x^k,$$

whenever the first series converges absolutely.

Let us verify that conditions (20) hold. Set $c_k = (1+k)^\gamma$.
Since, for $i \in \mathbb{N}$ and $k \in \mathbb{N}_0$, $0 < c_k \le c_{k+1}$ and

$$\frac{c_{k+i}}{c_k} = \Big(\frac{k+1+i}{k+1}\Big)^\gamma = \Big(1 + \frac{i}{k+1}\Big)^\gamma \le 1 + \frac{i}{k+1} \le 1 + \frac{ic_{k+1}}{(k+1)c_k},$$

one has

$$0 < c_{k+i} - c_k \le \frac{ic_{k+1}}{k+1}.$$

On the other hand, by the mean value theorem, there exists s, $k+1 < s < k+i+1$ such that

$$c_{k+i} - c_k = \gamma i \frac{1}{s^{1-\gamma}} < \gamma i \frac{1}{(k+1)^{1-\gamma}} < \gamma i (k+1)^{\gamma} = \gamma i c_k.$$

Example 5. For a fixed $j \in \mathbb{N}$, $n \in \mathbb{N}$, $f \in C(I)$, and $x \in I$, define

$$C_{n,j}(f,x) = \frac{1}{c_{n,j}(x)} \sum_{k=0}^{\infty} \frac{n^k}{(k+j)!} f\left(\frac{k}{n}\right) x^k, \qquad c_{n,j}(x) = \sum_{k=0}^{\infty} \frac{n^k}{(k+j)!} x^k,$$

whenever the first series converges absolutely.

In order to apply Theorem 7, we verify conditions (18) by considering, for each $k \in \mathbb{N}_0$, the decreasing sequence

$$\nu_{k,j} = \frac{1}{(k+1)\cdots(k+j)}. \tag{26}$$

First notice that, for any $m \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$,

$$\nu_{k+m,j} - \nu_{k+m+1,j} = \frac{1}{(k+m+2)\cdots(k+m+j)} \left(\frac{1}{(k+i+1)} - \frac{1}{(k+m+j+1)} \right)$$

$$= \frac{j}{(k+m+1)} \nu_{k+m+1,j} \le \frac{j}{(k+1)} \nu_{k+1,j}. \tag{27}$$

We will verify by induction with respect to i that, for $k \in \mathbb{N}_0$,

$$\nu_{k,j} - \nu_{k+i,j} \le \frac{ij}{(k+1)} \nu_{k+1,j}. \tag{28}$$

From (27) with $m = 0$, we know that

$$\nu_{k,j} - \nu_{k+1,j} \le \frac{j}{(k+1)} \nu_{k+1,j}.$$

This proved (28) for $i = 1$ and each $k \in \mathbb{N}_0$.

Assume that (28) holds for some $i \in \mathbb{N}$ and every $k \in \mathbb{N}_0$. If $k \in \mathbb{N}_0$, it follows (28) and (27) (with $m = i$) that

$$0 < \nu_{k,j} - \nu_{k+i+1,j} = \nu_{k,j} - \nu_{k+i,j} + \nu_{k+i,j} - \nu_{k+i+1,j}$$
$$\le \frac{ij}{(k+1)} \nu_{k+1,j} + \frac{j}{(k+1)} \nu_{k+1,j} \le \frac{(i+1)j}{(k+1)} \nu_{k+1,j}.$$

Example 6. For $0<\gamma<1$ fixed, $n\in\mathbb{N}$, $f\in C(I)$, and $x\in I$ define

$$E_{n,\gamma}(f,x)=\frac{1}{e_{n,\gamma}(x)}\sum_{k=0}^{\infty}\frac{n^k}{k!}\frac{1}{(k+1)^{\gamma}}f\Big(\frac{k}{n}\Big)x^k,\quad e_{n,\gamma}(x)=\sum_{k=0}^{\infty}\frac{n^k}{k!}\frac{1}{(k+1)^{\gamma}}x^k,$$

whenever the first series converges absolutely.

We apply Theorem 7 by considering, for $k\in\mathbb{N}_0$, the decreasing sequence $\delta_{k,\gamma}=(k+1)^{-\gamma}$. We will verify that

$$\delta_{k,\gamma}-\delta_{k+i,\gamma}\leq\frac{i\,\gamma\,\delta_{k+1,\gamma}}{k+1}.$$

Taking into account the mean value theorem, for each $i\in\mathbb{N}$, there exists

$$1<\theta<\frac{k+1+i}{k+1}$$

such that

$$(k+1)\Big(\delta_{k,\gamma}-\delta_{k+i,\gamma}\Big)=(k+1)\Big(\frac{1}{(k+1)^{\gamma}}-\frac{1}{(k+1+i)^{\gamma}}\Big)$$

$$=\frac{k+1}{(k+1+i)^{\gamma}}\Big(\frac{(k+1+i)^{\gamma}}{(k+1)^{\gamma}}-1\Big)=\frac{\gamma(k+1)}{(k+1+i)^{\gamma}}\Big(\frac{k+1+i}{k+1}-1\Big)\frac{1}{\theta^{1-\gamma}}$$

$$\leq\frac{\gamma(k+1)}{(k+1+i)^{\gamma}}\frac{i}{(k+1)}\leq\frac{i\gamma}{(k+2)^{\gamma}}=i\gamma\delta_{k+1,\gamma}.$$

Example 7. Let $a>0$ be a fixed real. For $k\in\mathbb{N}_0$, set $\sigma_{k,a}=e^{a/(k+1)}$. For $n\in\mathbb{N}$, $f\in C(I)$, and $x\in I$, define

$$D_{n,a}(f,x)=\frac{1}{d_{n,a}(x)}\sum_{k=0}^{\infty}\frac{\sigma_{k,a}}{k!}f\Big(\frac{k}{n}\Big)(nx)^k,\quad d_{n,a}(x)=\sum_{k=0}^{\infty}\frac{\sigma_{k,a}}{k!}(nx)^k,$$

whenever the first series converges absolutely.

We apply Theorem 7 by considering, for $k\in\mathbb{N}_0$, the decreasing sequence $\{\sigma_{k,a}\}$. We will verify by induction on i that there exists a constant $C(a,i)$ such that, for every $k\in\mathbb{N}_0$ and $i\in\mathbb{N}_0$, one has

$$\sigma_{k,a}-\sigma_{k+i,a}\leq\frac{C(a,i)}{(k+1)}\sigma_{k+1,a}.\tag{29}$$

Of course, for $i=0$, the previous inequality is trivial.

(i) We first estimate $\sigma_{k+j,a} - \sigma_{k+j+1,a}$, for any $j, k \in \mathbb{N}_0$. taking into account the mean valued theorem, there exists $\theta \in (0, a/(k+1+j) - a/(k+2+j))$ such that

$$\begin{aligned}\sigma_{k+j,a} - \sigma_{k+j+1,a} &= e^{a/(k+1+j)} - e^{a/(k+2+j)} = e^{a/(k+2+j)}(e^{a/(k+1+j)-1/(k+2+j)} - 1)\\ &= e^{a/(k+2+j)}\Big(\frac{a}{k+1+j} - \frac{a}{k+2+j}\Big)e^{\theta} \le \frac{ae^{a}\sigma_{k+1,a}}{(k+1)}.\end{aligned}$$

In particular, if $j = 0$, this proves (29) for $i = 1$.

(ii) If (29) holds for some $i \in \mathbb{N}$ and every $k \in \mathbb{N}_0$, by considering $j = i$ in the previous inequality, one has

$$\begin{aligned}\sigma_{k,a} - \sigma_{k+i+1,a} &= \sigma_{k,a} - \sigma_{k+i,a} + \sigma_{k+i,a} - \sigma_{k+i+1,a}\\ &\le \frac{C(a,i)}{(k+1)}\sigma_{k+1,a} + \frac{ae^{a}}{(k+1)}\sigma_{k+1,a} = \frac{C(a,i) + ae^{a}}{(k+1)}\sigma_{k+1,a}.\end{aligned}$$

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