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ON BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT DELAY

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ABSTRACT. The paper deals with establishing existence of solution to a boundary value problem of second order ordinary differential equation with constant delay of the type $y''(t) = f(t, y(t), y(t-r), y'(t))$. We use Banach fixed point theorem to establish the existence and uniqueness theorem by assuming f to be Lipschitz in second, third and fourth argument. We also use Schauder's fixed point theorem to prove the existence of solution for the boundary value problem by assuming f to be bounded.

1. Introduction. The study of boundary value problems (BVPs) for differential equations with deviating arguments (DEDA) is important because of their applications in various fields such as production problems in Economics, Biological systems, Physical models, optimal control theory etc. Thus, given a DEDA the most natural question that arises is of existence of its solution. The existence theory of ordinary differential equations as well as delay differential

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equations(DDE) is well developed in ([5, 6, 7]). For the fixed point theory, which is a main source of proving many existence results, one may refer to [15]. For BVPs of first order DDE refer [1, 8, 9, 13, 16, 18]. Results on positive solutions for boundary value problems of second order differential equations and delay differential equations are well established in [2, 4, 14]. A fixed point iterative method for the solution of two-point BVPs given by

$$\begin{aligned} y''(x) &= f(x, y(x), y'(x)), \quad a \leq x \leq b, \\ \lambda_0 y(a) + \beta_0 y'(a) &= \gamma_0, \quad \lambda_1 y(b) + \beta_1 y'(b) = \gamma_1, \end{aligned}$$

where λ_j, β_j are real numbers and $\lambda_j^2 + \beta_j^2 > 0$, $j = 0, 1$ is found in [3]. In [10], existence of ω -periodic solutions for second order delay differential equation $-u''(t) + b(t)u(t) = g(t)f(t, u(t), u(t - \tau))$, $t \in \mathbb{R}$ in Banach spaces is studied. BVPs of quadratic mixed type of delay differential equations given by

$$\begin{aligned} -u''(t) &= \lambda p(t)f \left[t, u(t - \tau), \int_0^t k(t, s)u(s)ds \right], \quad t \in (0, 1), \\ u(t) &= 0, \quad -\tau \leq t \leq 0 \quad \text{and} \quad u(1) = au(\eta) \end{aligned}$$

are investigated in [12]. In [19], the BVP for functional differential equation of second order of the type

$$\begin{aligned} x''(t) &= f(t, x_t), \quad t \in J = [0, T], T > 0, \\ x_0 &= \phi, \quad x'(T) = \beta x'(0), \beta > 1, \end{aligned}$$

where $f : J \times C([- \tau, 0], \mathbb{R}) \rightarrow \mathbb{R}$ is a given function, $\phi \in C([- \tau, 0], \mathbb{R})$, $\tau > 0$ is studied. For applications of second order DDE in optimal control theory one may refer [11] and [17].

In this paper we consider existence of solution to BVPs for a second order ordinary differential equation with constant Delay (second order DECD) using Banach fixed point theorem and Schauder's fixed point theorem.

Let $a, b \in \mathbb{R}$, $T > 0$, $I = [0, T]$, $0 < r \leq T$ and $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous. Then the BVP under investigation is given by,

$$(1.1) \quad y''(t) = f(t, y(t), y(t - r), y'(t)), \quad t \in I,$$

$$(1.2) \quad y(t) = a, \quad t \in [-r, 0],$$

$$(1.3) \quad y(T) + y'(T) = b.$$

We now state Arzela–Ascoli theorem, Schauder's fixed point theorem [15] and Banach contraction mapping theorem.

Theorem 1.1 (Arzela-Ascoli Theorem). *Let $\{f_n\}$ be uniformly bounded and uniformly equicontinuous sequence of functions defined on a non-empty subset E of $\mathbb{R}$. Then $\{f_n\}$ has a subsequence which is uniformly convergent on E .*

Theorem 1.2 (Schauder's Fixed Point Theorem). *Let E be a normed linear space, $S : E \rightarrow E$ be a continuous and compact map such that $S(E)$ is bounded. Then S has a fixed point.*

Theorem 1.3 (Banach Fixed Point Theorem). *Let (Y, d) be a complete metric space and $J : Y \rightarrow Y$ be a contraction. Then J has a unique fixed point in Y .*

2. Existence theorem using Banach fixed point theorem. In this section, we establish existence of unique solution to the BVP (1.1)–(1.3) using Banach fixed point theorem.

Definition 2.1. *A function $u : [-r, T] \rightarrow \mathbb{R}$ is said to be a solution of BVP (1.1) – (1.3) if,*

1. $u \in C[-r, T]$ and $u \in C^2[0, T]$.
2. $u''(t) = f(t, u(t), u(t-r), u'(t)), t \in I$.
3. $u(t) = a, t \in [-r, 0]$.
4. $u(T) + u'(T) = b$.

We need the following lemma, proof of which is straightforward and hence omitted.

Lemma 2.1. *y is a solution of the BVP (1.1)–(1.3) iff y satisfies*

$$(2.1) \quad y(t) = \begin{cases} a, & -r \leq t \leq 0, \\ a + \frac{(b-a)t}{1+T} + \int_0^T G(t,s)F(s;y)ds, & 0 \leq t \leq T, \end{cases}$$

$$(2.2) \quad \text{where } G(t,s) = \begin{cases} \frac{st}{1+T} - s, & 0 \leq s \leq t \leq T, \\ \frac{st}{1+T} - t, & 0 \leq t \leq s \leq T. \end{cases}$$

Let B be the set of all functions defined and continuous on $[-r, T]$ and continuously differentiable on $[0, T]$. Let $S = \{x \in B : x(t) = a, \forall t \in [-r, 0]\}$. We define $\|x\|_\infty = \sup_{t \in [-r, T]} |x(t)| + \sup_{t \in [0, T]} |x'(t)|$, $\forall x \in B$ and $d_\infty(x, y) =$

$\|x - y\|, \forall x, y \in B$. Then (B, d_∞) is a complete metric space and S is a closed subspace of (B, d_∞) . Hence (S, d_∞) is complete metric space itself.

Theorem 2.2. *Let $\mu_1 > 0, \mu_2 > 0, \mu_3 > 0, \mu = \max\{\mu_1, \mu_2, \mu_3\}$ such that $6\mu T(2T + 1) + 9\mu T < 1$. Let $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous and satisfy $|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq \mu_1|x_1 - x_2| + \mu_2|y_1 - y_2| + \mu_3|z_1 - z_2|, \forall (t, x_1, y_1, z_1), (t, x_2, y_2, z_2) \in I \times \mathbb{R}^3$. Then, the BVP (1.1) – (1.3) has a unique solution $y \in S$.*

Proof. Let $L_1 = 6\mu T(2T + 1)$ and $L_2 = 9\mu T$. Then $L = L_1 + L_2 < 1$. Let $J : S \rightarrow S$ be a mapping defined as

$$(2.3) \quad J[y(t)] = \begin{cases} a, & -r \leq t \leq 0, \\ a + \frac{(b-a)t}{1+T} + \int_0^T G(t, s)F(s; y)ds, & 0 \leq t \leq T, \end{cases}$$

where $G(t, s)$ is given by (2.2) and $F(t; y) = f(t, y(t), y(t-r), y'(t)), \forall t \in I$.

For $0 \leq t \leq T$, we may also express $J[y(t)]$ as

$$J[y(t)] = a + \frac{(b-a)t}{1+T} + \int_0^T \left[\frac{st}{1+T} - t \right] F(s; y)ds + \int_0^t (t-s)F(s; y)ds.$$

Therefore

$$J'[y(t)] = \frac{(b-a)}{1+T} + \int_0^T \left[\frac{s}{1+T} - 1 \right] F(s; y)ds + \int_0^t F(s; y)ds.$$

We show that $J : S \rightarrow S$ is a contraction.

Let $w, z \in S$ be arbitrary. Let $-r \leq t \leq 0$. Then $|J[w(t)] - J[z(t)]| = |a - a| = 0 \leq L_1 d_\infty(w, z)$.

Let $0 \leq t \leq T$. Then

$$\begin{aligned}
& |J[w(t)] - J[z(t)]| \\
& \leq \int_0^T |G(t, s)| |f(s, w(s), w(s-r), w'(s)) - f(s, z(s), z(s-r), z'(s))| ds \\
& \leq \int_0^T |G(t, s)| \{ \mu_1 |w(s) - z(s)| + \mu_2 |w(s-r) - z(s-r)| + \mu_3 |w'(s) - z'(s)| \} ds \\
& \leq 3\mu d_\infty(w, z) \int_0^T |G(t, s)| ds \\
& = 3\mu d_\infty(w, z) \left\{ \int_0^t |G(t, s)| ds + \int_t^T |G(t, s)| ds \right\} \\
& = \frac{3\mu d_\infty(w, z)}{T+1} \left\{ \int_0^t (st + sT + s) ds + \int_t^T (st + tT + t) ds \right\} \\
& \leq \frac{3\mu d_\infty(w, z)}{T} \left\{ \int_0^T (2T^2 + T) ds + \int_0^T (2T^2 + T) ds \right\} \\
& = 6\mu d_\infty(w, z) \int_0^T (2T + 1) ds = 6\mu T(2T + 1) d_\infty(w, z) = L_1 d_\infty(w, z).
\end{aligned}$$

Therefore

$$(2.4) \quad \sup_{t \in [-r, T]} |J[w(t)] - J[z(t)]| \leq L_1 d_\infty(w, z).$$

For $0 \leq t \leq T$, consider

$$\begin{aligned}
& |J'[w(t)] - J'[z(t)]| \\
& \leq \int_0^T \left| \frac{s}{1+T} - 1 \right| |f(s, w(s), w(s-r), w'(s)) - f(s, z(s), z(s-r), z'(s))| ds \\
& \quad + \int_0^t |f(s, w(s), w(s-r), w'(s)) - f(s, z(s), z(s-r), z'(s))| ds
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^T \left(\frac{s}{1+T} + 1 \right) \{ \mu_1 |w(s) - z(s)| + \mu_2 |w(s-r) - z(s-r)| + \mu_3 |w'(s) - z'(s)| \} ds \\
&\quad + \int_0^t \{ \mu_1 |w(s) - z(s)| + \mu_2 |w(s-r) - z(s-r)| + \mu_3 |w'(s) - z'(s)| \} ds \\
&\leq 3\mu d_\infty(w, z) \int_0^T \left(\frac{s}{T} + 1 \right) ds + 3\mu d_\infty(w, z) \int_0^T ds \\
&\leq 6\mu d_\infty(w, z) \int_0^T ds + 3\mu d_\infty(w, z) \int_0^T ds = 9\mu T d_\infty(w, z) = L_2 d_\infty(w, z).
\end{aligned}$$

Therefore

$$(2.5) \quad \sup_{t \in [-r, T]} |J'[w(t)] - J'[z(t)]| \leq L_2 d_\infty(w, z).$$

Adding inequalities (2.4) and (2.5), we get $d_\infty(Jw, Jz) \leq L d_\infty(w, z)$, $\forall w, z \in S$ and $L < 1$. Therefore $J : S \rightarrow S$ is a contraction mapping. Thus J has a unique fixed point $y \in X$. Hence y is a unique solution of the BVP (1.1)–(1.3). $\square$

Next let $\gamma, \theta \in \mathbb{R}$. Consider the BVP given by

$$\begin{aligned}
(2.6) \quad &y''(t) + \gamma y'(t) + \theta y(t) = f(t, y(t), y(t-r), y'(t)), \quad t \in I, \\
&y(t) = a, \quad t \in [-r, 0], \\
&y(T) + y'(T) = b.
\end{aligned}$$

We now establish the existence and uniqueness of solution to the above BVP as a corollary to Theorem 2.2.

Corollary 2.3. *Let $\mu_1 > 0$, $\mu_2 > 0$, $\mu_3 > 0$, $\mu = \max\{\mu_1 + |\theta|, \mu_2, \mu_3 + |\gamma|\}$ such that $6\mu T(2T + 1) + 9\mu T < 1$. Let $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous and satisfy $|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq \mu_1 |x_1 - x_2| + \mu_2 |y_1 - y_2| + \mu_3 |z_1 - z_2|$, $\forall (t, x_1, y_1, z_1), (t, x_2, y_2, z_2) \in I \times \mathbb{R}^3$. Then the BVP (2.6), (1.2) and (1.3) has a unique solution.*

Proof. Let $F^*(t, u, v, w) = f(t, u, v, w) - \theta u - \gamma w$, $\forall (t, u, v, w) \in I \times \mathbb{R}^3$. Consider the BVP given by,

$$(2.7) \quad y''(t) = F^*(t, y(t), y(t-r), y'(t)), \quad t \in I,$$

$$\begin{aligned} y(t) &= a, \quad t \in [-r, 0], \\ y(T) + y'(T) &= b. \end{aligned}$$

Let $(t, u_1, v_1, w_1), (t, u_2, v_2, w_2) \in I \times \mathbb{R}^3$ be arbitrary. Then

$$\begin{aligned} &|F^*(t, u_1, v_1, w_1) - F^*(t, u_2, v_2, w_2)| \\ &= |f(t, u_1, v_1, w_1) - \theta u_1 - \gamma w_1 - f(t, u_2, v_2, w_2) + \theta u_2 + \gamma w_2| \\ &\leq |f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)| + |\theta||u_1 - u_2| + |\gamma||w_1 - w_2| \\ &\leq (\mu_1 + |\theta|)|u_1 - u_2| + \mu_2|v_1 - v_2| + (\mu_3 + |\gamma|)|w_1 - w_2|. \end{aligned}$$

Therefore, by applying Theorem 2.2, we conclude that the BVP (2.7), (1.2) and (1.3) has a unique solution y .

So we have

$$\begin{aligned} y''(t) &= f(t, y(t), y(t-r), y'(t)) - \theta y(t) - \gamma y'(t), \quad t \in I, \\ y(t) &= a, \quad t \in [-r, 0], \\ y(T) + y'(T) &= b. \end{aligned}$$

Hence y is the unique solution of the BVP (2.6), (1.2) and (1.3). $\square$

3. Existence theorem using Schauder's fixed point theorem.

In this section, we establish existence of a solution to the BVP (1.1) – (1.3) by using Schauder's fixed point theorem. We take $a = 0$, so as to satisfy conditions of the Schauder's fixed point theorem. Then our BVP is

$$\begin{aligned} (3.1) \quad &y''(t) = f(t, y(t), y(t-r), y'(t)), \quad t \in I, \\ &y(t) = 0, \quad t \in [-r, 0], \\ &y(T) + y'(T) = b. \end{aligned}$$

We need the following definition.

Definition 3.1. A function $u : [-r, T] \rightarrow \mathbb{R}$ is said to be a solution of the BVP (1.1), (3.1) and (1.3) if,

1. $u \in C[-r, T]$ and $u \in C^2[0, T]$.
2. $u''(t) = f(t, u(t), u(t-r), u'(t)), \quad t \in I$.
3. $u(t) = 0, \quad t \in [-r, 0]$.
4. $u(T) + u'(T) = b$.

Note that y is a solution of BVP (1.1), (3.1) and (1.3) iff y satisfies

$$(3.2) \quad y(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \frac{bt}{1+T} + \int_0^T G(t, s)F(s; y)ds, & 0 \leq t \leq T, \end{cases}$$

where $G(t, s)$ is given by (2.2) and $F(t; y) = f(t, y(t), y(t-r), y'(t))$, $\forall t \in I$.

Let $Y = \{x \in B : x(t) = 0, \forall t \in [-r, 0]\}$. Then $(B, \|\cdot\|_\infty)$ and $(Y, \|\cdot\|_\infty)$ are Banach spaces.

We break down our main theorem of this section into two lemmas.

Lemma 3.1. *Let $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous. Then $N : Y \rightarrow Y$ defined by*

$$N[y(t)] = \begin{cases} 0, & -r \leq t \leq 0, \\ \frac{bt}{1+T} + \int_0^T G(t, s)F(s; y)ds, & 0 \leq t \leq T, \end{cases}$$

is continuous.

Proof. For $0 \leq t \leq T$, we may express $N[y(t)]$ as,

$$N[y(t)] = \frac{bt}{1+T} + \int_0^T \left[\frac{st}{1+T} - t \right] F(s; y)ds + \int_0^t (t-s)F(s; y)ds.$$

Therefore

$$N'[y(t)] = \frac{b}{1+T} + \int_0^T \left[\frac{s}{1+T} - 1 \right] F(s; y)ds + \int_0^t F(s; y)ds.$$

Let (y_n) be a sequence in $(Y, \|\cdot\|_\infty)$ such that $y_n \rightarrow y$ in $(Y, \|\cdot\|_\infty)$. Then $\|y_n - y\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Therefore $y_n \rightarrow y$ uniformly on $[-r, T]$ and $y'_n \rightarrow y'$ uniformly on $[0, T]$, i.e. $\sup_{t \in [-r, T]} |y_n(t) - y(t)| \rightarrow 0$ as $n \rightarrow \infty$ and

$$\sup_{t \in [0, T]} |y'_n(t) - y'(t)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Let $v_n = Ny_n$ and $v = Ny$. We prove that $v_n \rightarrow v$ in $(Y, \|\cdot\|_\infty)$. We first show that $v_n \rightarrow v$ uniformly on $[-r, T]$.

Let $\epsilon > 0$ be arbitrary. Since $y_n \rightarrow y$ uniformly on $[-r, T]$, $y_n \rightarrow y$ uniformly on $[0, T]$. Therefore, $y_n(t-r) \rightarrow y(t-r)$ uniformly on $[0, T]$. This

implies $f(t, y_n(t), y_n(t-r), y'_n(t)) \rightarrow f(t, y(t), y(t-r), y'(t))$ uniformly on I as $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous. Therefore $\exists n_0 \in \mathbb{N}$ such that $|f(t, y_n(t), y_n(t-r), y'_n(t)) - f(t, y(t), y(t-r), y'(t))|, \forall n \geq n_0$ and $\forall t \in I$, i.e. $|F(t; y_n) - F(t; y)| < \frac{\epsilon}{8T^2}, \forall n \geq n_0$ and $\forall t \in I$.

Let $-r \leq t \leq 0$. Then

$$(3.3) \quad |v_n(t) - v(t)| = 0 < \epsilon, \quad \forall n \geq n_0 \text{ and } \forall t \in [-r, 0].$$

Let $0 \leq t \leq T$. Then

$$\begin{aligned} |v_n(t) - v(t)| &= \left| \int_0^T G(t, s) F(s; y_n) ds - \int_0^T G(t, s) F(s; y) ds \right| \\ &\leq \int_0^T |G(t, s)| |F(s; y_n) - F(s; y)| ds \\ &< \frac{\epsilon}{8T^2} \int_0^T |G(t, s)| ds \\ &= \frac{\epsilon}{8T^2} \left\{ \int_0^t |G(t, s)| ds + \int_t^T |G(t, s)| ds \right\} \\ &= \frac{\epsilon}{8T^2} \left\{ \int_0^t \left| \frac{st}{1+T} - s \right| ds + \int_t^T \left| \frac{st}{1+T} - t \right| ds \right\} \\ &\leq \frac{\epsilon}{8T^2} \left\{ \int_0^T \left(\frac{st}{T} + s \right) ds + \int_0^T \left(\frac{st}{T} + t \right) ds \right\} \\ &\leq \frac{\epsilon}{8T^2} \left\{ \int_0^T 2T ds + \int_0^T 2T ds \right\} = \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Therefore

$$(3.4) \quad |v_n(t) - v(t)| < \epsilon, \quad \forall n \geq n_0 \text{ and } \forall t \in [0, T].$$

Thus from (3.3) and (3.4), we conclude that, $|v_n(t) - v(t)| < \epsilon, \forall n \geq n_0$ and $\forall t \in [-r, T]$. This implies $v_n \rightarrow v$ uniformly on $[-r, T]$.

We next show that $v'_n \rightarrow v'$ uniformly on $[0, T]$. Let $\epsilon > 0$ be arbitrary. Since $F(t; y_n) \rightarrow F(t; y)$ uniformly on I , $\exists n_0 \in \mathbb{N}$ such that $|F(t; y_n) - F(t; y)| < \frac{\epsilon}{8T}$, $\forall n \geq n_0$ and $\forall t \in [0, T]$.

Let $0 \leq t \leq T$. Then

$$\begin{aligned} |v'_n(t) - v'(t)| &\leq \int_0^T \left| \frac{s}{1+T} - 1 \right| |F(s; y_n) - F(s; y)| ds + \int_0^t |F(s; y_n) - F(s; y)| ds \\ &\leq \int_0^T \left(\frac{s}{T} + 1 \right) |F(s; y_n) - F(s; y)| ds + \int_0^t |F(s; y_n) - F(s; y)| ds \\ &\leq 2 \int_0^T |F(s; y_n) - F(s; y)| ds + \int_0^T |F(s; y_n) - F(s; y)| ds \\ &< \frac{\epsilon}{4T} \int_0^T ds + \frac{\epsilon}{8T} \int_0^T ds = \frac{3\epsilon}{8} < \epsilon. \end{aligned}$$

Therefore $|v'_n(t) - v'(t)| < \epsilon$, $\forall n \geq n_0$ and $\forall t \in [0, T]$. This implies $v'_n \rightarrow v'$ uniformly on $[0, T]$. So we have $v_n \rightarrow v$ uniformly on $[-r, T]$ and $v'_n \rightarrow v'$ uniformly on $[0, T]$. Therefore $v_n \rightarrow v$ on $(Y, \|\cdot\|_\infty)$, i.e. $Ny_n \rightarrow Ny$ on $(Y, \|\cdot\|_\infty)$. Hence $N : Y \rightarrow Y$ is continuous. $\square$

Lemma 3.2. Let $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous and bounded. Let $N : Y \rightarrow Y$ be defined by

$$N[y(t)] = \begin{cases} 0, & -r \leq t \leq 0, \\ \frac{bt}{1+T} + \int_0^T G(t, s)F(s; y)ds, & 0 \leq t \leq T. \end{cases}$$

Then $N(Y)$ is bounded subset of $(Y, \|\cdot\|_\infty)$ and $N : Y \rightarrow Y$ is compact.

Proof. Since $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is bounded, $\exists M > 0$ such that

$$|f(t, u, v, w)| \leq M, \quad \forall (t, u, v, w) \in I \times \mathbb{R}^3.$$

For $0 \leq t \leq T$, we can express $N[y(t)]$ as

$$N[y(t)] = \frac{bt}{1+T} + \int_0^T \left[\frac{st}{1+T} - t \right] F(s; y) ds + \int_0^t (t-s) F(s; y) ds.$$

Therefore

$$N'[y(t)] = \frac{b}{1+T} + \int_0^T \left[\frac{s}{1+T} - 1 \right] F(s; y) ds + \int_0^t F(s; y) ds.$$

We first show that $N(Y)$ is a bounded subset of $(Y, \|\cdot\|_\infty)$.

Let $y \in Y$ be arbitrary. If $-r \leq t \leq 0$, then $|N[y(t)]| = 0 \leq |b| + 4MT^2$.
Let $0 \leq t \leq T$. Then

$$\begin{aligned} |N[y(t)]| &\leq \frac{|b||t|}{1+T} + \int_0^T \left| \frac{st}{1+T} - t \right| |F(s; y)| ds + \int_0^t |t-s| |F(s; y)| ds \\ &\leq |b| + \int_0^T \left(\frac{st}{T} + t \right) |F(s; y)| ds + \int_0^T (t+s) |F(s; y)| ds \\ &\leq |b| + 2TM \int_0^T ds + 2TM \int_0^T ds = |b| + 4MT^2. \end{aligned}$$

Therefore $|N[y(t)]| \leq |b| + 4MT^2$, $\forall t \in [-r, T]$ and $\sup_{t \in [-r, T]} |N[y(t)]| \leq |b| + 4MT^2$.

Now for $0 \leq t \leq T$, we have

$$\begin{aligned} |N'[y(t)]| &\leq \frac{|b|}{1+T} + \int_0^T \left| \frac{s}{1+T} - 1 \right| |F(s; y)| ds + \int_0^t |F(s; y)| ds \\ &\leq \frac{|b|}{T} + \int_0^T \left(\frac{s}{T} + 1 \right) |F(s; y)| ds + \int_0^T |F(s; y)| ds \\ &\leq \frac{|b|}{T} + 2M \int_0^T ds + M \int_0^T ds = \frac{|b|}{T} + 3MT. \end{aligned}$$

Therefore $|N'[y(t)]| \leq \frac{|b|}{T} + 3MT$, $\forall t \in [0, T]$ and $\sup_{t \in [0, T]} |N'[y(t)]| \leq \frac{|b|}{T} + 3MT$.

So we have

$$\|Ny\|_\infty = \sup_{t \in [-r, T]} |N[y(t)]| + \sup_{t \in [0, T]} |N'[y(t)]| \leq |b| + 4MT^2 + \frac{|b|}{T} + 3MT.$$

Hence $\|Ny\|_\infty \leq |b| + 4MT^2 + \frac{|b|}{T} + 3MT$, $\forall y \in Y$. This shows that $N(Y)$ is bounded subset of $(Y, \|\cdot\|_\infty)$.

Next we show that $N : Y \rightarrow Y$ is a compact operator. Let (y_n) be a bounded sequence in $(Y, \|\cdot\|_\infty)$ and let $v_n = Ny_n$. Since $N(Y)$ is a bounded subset of $(Y, \|\cdot\|_\infty)$, (v_n) is a bounded sequence in $(Y, \|\cdot\|_\infty)$. Therefore (v_n) is uniformly bounded on $[-r, T]$ and (v'_n) is uniformly bounded on $[0, T]$.

Claim 1. (v_n) is equicontinuous on $[-r, T]$.

Let $t_0 \in [-r, T]$ and $\epsilon > 0$ be arbitrary.

Case 1.1. Let $t_0 = -r$. Choose $\delta = \min \left\{ \frac{r}{2}, \epsilon \right\}$. Then, for $-r \leq t < -r + \delta$, $|v_n(t) - v_n(t_0)| = 0 < \epsilon$. This implies (v_n) is equicontinuous at $t_0 = -r$.

Case 1.2. Let $t_0 = 0$. Choose $\delta = \min \left\{ \frac{r}{2}, \frac{T}{|b| + 4MT^2} \epsilon \right\}$. Let t be such that $|t| < \delta$. If $t \leq t_0$ then $|v_n(t) - v_n(t_0)| = 0 < \epsilon$. Let $t \geq t_0$. Then

$$\begin{aligned}
 & |v_n(t) - v_n(t_0)| \\
 & \leq \frac{|b||t|}{1+T} + \int_0^T \left| \frac{st}{1+T} - t \right| |F(s; y_n)| ds + \int_0^t |t-s| |F(s; y_n)| ds \\
 & \leq \frac{|b||t|}{T} + |t| \int_0^T \left(\frac{s}{T} + 1 \right) |F(s; y_n)| ds + \int_0^t (t+s) |F(s; y_n)| ds \\
 & \leq \frac{|b||t|}{T} + 2M|t| \int_0^T ds + 2TM \int_0^t ds \\
 & = \frac{|b||t|}{T} + 4MT|t| < \left(\frac{|b| + 4MT^2}{T} \right) \delta \\
 & \leq \left(\frac{|b| + 4MT^2}{T} \right) \left(\frac{T}{|b| + 4MT^2} \right) \epsilon = \epsilon.
 \end{aligned}$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall t$ such that $|t| < \delta$. This implies (v_n) is equicontinuous at $t_0 = 0$.

Case 1.3. Let $t_0 = T$. Choose $\delta = \min \left\{ \frac{T}{2}, \frac{T}{|b| + 5MT^2} \epsilon \right\}$. Let t be such that $T - \delta < t \leq T$. Then

$$\begin{aligned}
 & |v_n(t) - v_n(t_0)| \\
 & = |v_n(T) - v_n(t)|
 \end{aligned}$$

$$\begin{aligned}
&= \left| \frac{b(T-t)}{1+T} + \int_0^T (T-t) \left[\frac{s}{1+T} - 1 \right] F(s; y_n) ds \right| \\
&\quad + \left| \int_0^T (T-s) F(s; y_n) ds - \int_0^t (t-s) F(s; y_n) ds \right| \\
&\leq \frac{|b|(T-t)}{T} + (T-t) \int_0^T \left(\frac{s}{T} + 1 \right) |F(s; y_n)| ds \\
&\quad + \left| \int_0^T (T-s) F(s; y_n) ds + \int_t^0 (t-s) F(s; y_n) ds \right| \\
&\leq \frac{|b|(T-t)}{T} + 2MT(T-t) \\
&\quad + \left| \int_0^T TF(s; y_n) ds + \int_t^0 tF(s; y_n) ds - \int_t^T sF(s; y_n) ds \right| \\
&\leq \frac{|b|(T-t)}{T} + 2MT(T-t) \\
&\quad + \left| \int_0^T TF(s; y_n) ds + \int_t^0 tF(s; y_n) ds \right| + \int_t^T s|F(s; y_n)| ds \\
&\leq \frac{|b|(T-t)}{T} + 3MT(T-t) + \left| \int_0^T TF(s; y_n) ds + \int_t^0 tF(s; y_n) ds \right| \\
&= \frac{|b|(T-t)}{T} + 3MT(T-t) \\
&\quad + \left| \int_0^T TF(s; y_n) ds - \int_0^T tF(s; y_n) ds + \int_0^T tF(s; y_n) ds + \int_t^0 tF(s; y_n) ds \right| \\
&= \frac{|b|(T-t)}{T} + 3MT(T-t) + \left| \int_0^T (T-t)F(s; y_n) ds + \int_t^T tF(s; y_n) ds \right| \\
&\leq \frac{|b|(T-t)}{T} + 3MT(T-t) + M(T-t) \int_0^T ds + MT \int_t^T ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|b|(T-t)}{T} + 5MT(T-t) \\
&< \left(\frac{|b| + 5MT^2}{T} \right) \delta \\
&\leq \left(\frac{|b| + 5MT^2}{T} \right) \left(\frac{T}{|b| + 5MT^2} \right) \epsilon = \epsilon.
\end{aligned}$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall t$ such that $T - \delta < t \leq T$. This implies (v_n) is equicontinuous at $t_0 = T$.

Case 1.4. Let $-r < t_0 < 0$. Choose $\delta = \min \left\{ t_0 + \frac{r}{2}, -\frac{t_0}{2} \right\}$. Then $|v_n(t) - v_n(t_0)| = 0 < \epsilon$, $\forall t$ such that $|t - t_0| < \epsilon$. This implies (v_n) is equicontinuous at t_0 , where $-r < t_0 < 0$.

Case 1.5. Let $0 < t_0 < T$. Choose $\delta = \min \left\{ \frac{t_0}{2}, \frac{T - t_0}{2}, \frac{T}{|b| + 5MT^2} \epsilon \right\}$. Let t be such that $|t - t_0| < \delta$. Then

$$\begin{aligned}
&|v_n(t) - v_n(t_0)| \\
&\leq \left| \frac{b(t - t_0)}{1 + T} + \int_0^T (t - t_0) \left[\frac{s}{1 + T} - 1 \right] F(s; y_n) ds \right| \\
&\quad + \left| \int_0^t (t - s) F(s; y_n) ds - \int_0^{t_0} (t_0 - s) F(s; y_n) ds \right| \\
&\leq \frac{|b||t - t_0|}{T} + |t - t_0| \int_0^T \left(\frac{s}{T} + 1 \right) |F(s; y_n)| ds \\
&\quad + \left| \int_0^t (t - s) F(s; y_n) ds + \int_{t_0}^0 (t_0 - s) F(s; y_n) ds \right| \\
&\leq \frac{|b||t - t_0|}{T} + 2MT|t - t_0| \\
&\quad + \left| \int_0^t t F(s; y_n) ds + \int_{t_0}^0 t_0 F(s; y_n) ds \right| + \left| \int_{t_0}^t s |F(s; y_n)| ds \right| \\
&\leq \frac{|b||t - t_0|}{T} + 3MT|t - t_0|
\end{aligned}$$

$$\begin{aligned}
& + \left| \int_0^t tF(s; y_n)ds - \int_0^t t_0F(s; y_n)ds + \int_0^t t_0F(s; y_n)ds + \int_{t_0}^0 t_0F(s; y_n)ds \right| \\
& = \frac{|b||t-t_0|}{T} + 3MT|t-t_0| + \left| \int_0^t (t-t_0)F(s; y_n)ds + \int_{t_0}^t t_0F(s; y_n)ds \right| \\
& \leq \frac{|b||t-t_0|}{T} + 3MT|t-t_0| + |t-t_0| \int_0^T |F(s; y_n)|ds + \left| \int_{t_0}^t t_0F(s; y_n)ds \right| \\
& \leq \frac{|b||t-t_0|}{T} + 3MT|t-t_0| + M|t-t_0| \int_0^T ds + MT \left| \int_{t_0}^t ds \right| \\
& \leq \frac{|b||t-t_0|}{T} + 5MT|t-t_0| \\
& < \left(\frac{|b| + 5MT^2}{T} \right) \delta \\
& \leq \left(\frac{|b| + 5MT^2}{T} \right) \left(\frac{T}{|b| + 5MT^2} \right) \epsilon = \epsilon.
\end{aligned}$$

Therefore $|v_n(t) - v_n(t_0)| < \epsilon$, $\forall t$ such that $|t - t_0| < \delta$. This implies (v_n) is equicontinuous at t_0 , where $0 < t_0 < T$.

Hence, from case 1.1 – case 1.5, we may conclude that (v_n) is equicontinuous on $[-r, T]$ uniformly.

Now using Arzela–Ascoli theorem, (v_n) has a uniformly convergent subsequence $(v_{(n,m)})$ converging uniformly to v on $[-r, T]$, i.e $v_{(n,m)} \rightarrow v$ uniformly $[-r, T]$ as $m \rightarrow \infty$.

Since each $v_{(n,m)}$ is continuous on $[-r, T]$, v is continuous on $[-r, T]$. Since (v_n) and (v'_n) are uniformly bounded on $[-r, T]$ and $[0, T]$, respectively, $(v_{(n,m)})$ and $(v'_{(n,m)})$ are uniformly bounded on $[-r, T]$ and $[0, T]$, respectively.

Let $z_m = v'_{(n,m)}$.

Claim 2. (z_m) is equicontinuous on $[0, T]$ uniformly. Let $t_0 \in [0, T]$ be arbitrary and $\epsilon > 0$ be arbitrary.

Case 2.1. Let $t_0 = 0$. Choose $\delta = \min\{\frac{T}{2}, \frac{\epsilon}{M}\}$. Let t be such that $0 \leq t < \delta$. Then

$$\begin{aligned}
|z_m(t) - z_m(t_0)| &= \left| \int_0^t F(s; y_{(n,m)}) ds \right| \leq \int_0^t |F(s; y_{(n,m)})| ds \\
&\leq M \int_0^t ds = tM < M\delta < \epsilon.
\end{aligned}$$

Thus $|z_m(t) - z_m(t_0)| < \epsilon$, $\forall t$ such that $0 \leq t < \delta$. Therefore (z_m) is equicontinuous at $t_0 = 0$.

Case 2.2. Let $t_0 = T$. Choose $\delta = \min\{\frac{T}{2}, \frac{\epsilon}{M}\}$. Let t be such that $T - \delta < t \leq T$. Then

$$\begin{aligned}
|z_m(t) - z_m(t_0)| &= |z_m(T) - z_m(t)| \\
&= \left| \int_0^T F(s; y_{(n,m)}) ds - \int_0^t F(s; y_{(n,m)}) ds \right| \\
&\leq \int_t^T |F(s; y_{(n,m)})| ds \\
&\leq M \int_t^T ds = (T - t)M < M\delta \leq \epsilon.
\end{aligned}$$

Thus $|z_m(t) - z_m(t_0)| < \epsilon$, $\forall t$ such that $T - \delta < t \leq T$. Therefore (z_m) is equicontinuous at $t_0 = T$.

Case 2.3. Let $0 < t_0 < T$. Choose $\delta = \min\{\frac{t_0}{2}, \frac{T - t_0}{2}, \frac{\epsilon}{M}\}$. Let t be such that $|t - t_0| < \delta$. Then

$$\begin{aligned}
|z_m(t) - z_m(t_0)| &= \left| \int_0^t F(s; y_{(n,m)}) ds - \int_0^{t_0} F(s; y_{(n,m)}) ds \right| \\
&\leq \int_{t_0}^t |F(s; y_{(n,m)})| ds \\
&\leq M \int_{t_0}^t ds = M|t - t_0| < M\delta \leq \epsilon.
\end{aligned}$$

Therefore $|z_m(t) - z_m(t_0)| < \epsilon$, $\forall t$ such that $|t - t_0| < \delta$.

This implies (z_m) is equicontinuous at t_0 , where $0 < t_0 < T$. Hence, from case 2.1 – case 2.3, we conclude that (z_m) is equicontinuous on $[0, T]$ uniformly, i.e. $(v'_{(n,m)})$ is equicontinuous on $[0, T]$ uniformly.

Again making use of Arzela–Ascoli theorem, we may find a subsequence $(v'_{(n,m,p)})$ of $(v'_{(n,m)})$ converging uniformly to u on $[0, T]$, i.e. $v'_{(n,m,p)} \rightarrow u$ uniformly on $[0, T]$ as $p \rightarrow \infty$. Since $v_{(n,m)} \rightarrow v$ uniformly on $[-r, T]$ as $m \rightarrow \infty$, $v_{(n,m)} \rightarrow v$ uniformly on $[0, T]$ as $m \rightarrow \infty$. As $(v'_{(n,m,p)})$ is a subsequence of $(v_{(n,m)})$, we may conclude that $v_{(n,m,p)} \rightarrow v$ uniformly on $[0, T]$ as $p \rightarrow \infty$. Therefore $v_{(n,m,p)} \rightarrow v$ uniformly on $[0, T]$ as $p \rightarrow \infty$ and $v_{(n,m,p)'} \rightarrow u$ uniformly on $[0, T]$ as $p \rightarrow \infty$ implies that v is differentiable on $[0, T]$ and $v' = u$ on $[0, T]$. Thus $v_{(n,m,p)} \rightarrow v$ uniformly on $[-r, T]$ as $p \rightarrow \infty$ and $v_{(n,m,p)'} \rightarrow v'$ uniformly on $[0, T]$ as $p \rightarrow \infty$, i.e. $\|v_{(n,m,p)} - v\|_\infty \rightarrow 0$ as $p \rightarrow \infty$. Also $(v_{(n,m,p)}) \subseteq Y$ and $(Y, \|\cdot\|_\infty)$ is a Banach space implies that $v \in Y$. Therefore $v_{(n,m,p)} \rightarrow v$ in $(Y, \|\cdot\|_\infty)$. Thus (Ny_n) has a subsequence $(Ny_{(n,m,p)})$ converging in $(Y, \|\cdot\|_\infty)$. Hence $N : Y \rightarrow Y$ is a compact operator. $\square$

We now combine Lemma 3.1 and Lemma 3.2 to get the following result on existence of solution.

Theorem 3.3. *Let $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be continuous and bounded. Then the BVP (1.1), (3.1) and (1.3) has at least one solution.*

Proof. Let $N : Y \rightarrow Y$ be defined as

$$N[y(t)] = \begin{cases} 0, & -r \leq t \leq 0, \\ \frac{bt}{1+T} + \int_0^T G(t,s)F(s;y)ds, & 0 \leq t \leq T, \end{cases}$$

where $G(t,s)$ is given by (2.2) and $F(t;y) = f(t, y(t), y(t-r), y'(t))$, $\forall t \in I$.

Since $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous, $N : Y \rightarrow Y$ is continuous. (By Lemma 3.1) Since $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and bounded, $N(Y)$ is bounded subset of $(Y, \|\cdot\|_\infty)$ and $N : Y \rightarrow Y$ is compact. (By Lemma 3.2) Therefore by applying Schauder's fixed point theorem, $N : Y \rightarrow Y$ has a fixed point $x \in Y$. Thus x satisfies $x(t) = 0, t \in [-r, 0]$, $x(T) + x'(T) = b$ and

$$x(t) = \frac{bt}{1+T} + \int_0^T \left[\frac{st}{1+T} - t \right] F(s;x)ds + \int_0^t (t-s)F(s;x)ds, \quad t \in [0, T].$$

Therefore

$$x'(t) = \frac{b}{1+T} + \int_0^T \left[\frac{s}{1+T} - 1 \right] F(s; x) ds + \int_0^t F(s; x) ds, \quad t \in [0, T].$$

Since $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous, x' is differentiable on $[0, T]$ and $x''(t) = f(t, x(t), x(t-r), x'(t))$, $t \in [0, T]$. Thus $x \in C[-r, T]$ and $x \in C^2[0, T]$. Hence x is a solution of the BVP (1.1), (3.1) and (1.3). $\square$

We illustrate the above theorem by following example.

Example 3.1. Consider the boundary value problem.

$$\begin{aligned} x''(t) &= \cos \left[x \left(t - \frac{\pi}{2} \right) \right], \quad t \in I, \\ x(t) &= 0, \quad t \in [-r, 0], \\ x \left(\frac{\pi}{2} \right) + x' \left(\frac{\pi}{2} \right) &= 1. \end{aligned}$$

Here $r = T = \frac{\pi}{2}$, $I = \left[0, \frac{\pi}{2} \right]$, $f(t, u, v, w) = \cos v$, $\forall (t, u, v, w) \in I \times \mathbb{R}^3$ and $b = 1$. Clearly $f : I \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is bounded and continuous. Let $0 \leq t \leq \frac{\pi}{2}$. Then $-\frac{\pi}{2} \leq t - \frac{\pi}{2} \leq 0$. Therefore $x \left(t - \frac{\pi}{2} \right) = 0$, $\forall t \in \left[0, \frac{\pi}{2} \right]$. This implies $\cos \left[x \left(t - \frac{\pi}{2} \right) \right] = 1$, $\forall t \in \left[0, \frac{\pi}{2} \right]$. So $x''(t) = 1$, $\forall t \in \left[0, \frac{\pi}{2} \right]$. Thus $x(t) = x'(0)t + \frac{t^2}{2}$, $t \in \left[0, \frac{\pi}{2} \right]$ and $x'(t) = x'(0) + t$, $t \in \left[0, \frac{\pi}{2} \right]$. Substituting $t = \frac{\pi}{2}$ in above expressions of $x(t)$ and $x'(t)$ we get $x'(0) = \frac{8 - 4\pi - \pi^2}{8 + 4\pi}$. Hence $x(t) = \left(\frac{8 - 4\pi - \pi^2}{8 + 4\pi} \right) t + \frac{t^2}{2}$, $t \in \left[0, \frac{\pi}{2} \right]$ is a solution to the given BVP.

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