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A NEW TYPE OF FIXED POINT RESULTS FOR MODIFIED WEAK CONTRACTION IN CUBIC TERMS OF METRIC VIA SIMULATION FUNCTION

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ABSTRACT. In this manuscript, we introduce new notion of modified weak contraction in cubic terms of metric via simulation functions and establish fixed point results for this kind of operator in the framework of complete metric space. Our result is an extension of δ -weak contraction in complete metric space which generalize some existing results. Moreover, example is given to illustrate the attained sequel.

1. Introduction. Fixed point is very powerful tool in several spheres of analysis. In 2013, Murthy and Prasad [9] generalize weak contraction by making combinations of $\sigma(x, y)$. In 2015, Khojasteh et al. [7] established some fixed point results for \mathcal{Z} -contraction via simulation function. Afterwards, Argoubi et al. [1] introduced a contraction comprising simulation function for a metric enriched with partial order. In 2017, Jain et al. [6] proved some results for commuting mappings comprising cubic terms of metric. In 2020, Arora et al. conferred common fixed

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point results for modified β -admissible contraction and almost Z -contraction in the edge of metric space and G-metric space (see [3], [8]). Afterwards, Debnath et al. [4] discussed the existence and uniqueness of common fixed point theorems for Kannan, Reich and Chatterjea type pair of self-maps in the context of complete metric space. In 2021, Arora [2] established some common fixed point results for four self maps in the context of Gs-metric space via CLR property.

Now, let us recall some definitions which will be significant in the sequel.

Definition 1.1 ([7]). *Let (X, σ) be a metric space, f is a self-mapping on X and $\zeta \in \mathcal{Z}$. We say that f is a \mathcal{Z} -contraction with respect to ζ , if $\zeta(\sigma(fx, fy), \sigma(x, y)) \geq 0$, for all $x, y \in X$.*

Hang and Dung [5] investigated the concept of generalized F-contraction and proved useful fixed point results for such kind of functions.

Definition 1.2 ([5]). *Let (X, σ) be a metric space and $f : X \rightarrow X$ be a self-mapping. f is called a generalized F-contraction on (X, σ) if there exist $F \in \mathfrak{S}$ and $\delta > 0$ such that*

$$\sigma(fx, fy) > 0 \Rightarrow \delta + F(\sigma(fx, fy)) \leq F \left(\max \left\{ \sigma(x, y), \sigma(x, fx), \sigma(y, fy), \frac{\sigma(x, fy) + \sigma(y, fx)}{2}, \frac{\sigma(f^2x, x) + \sigma(f^2x, fy)}{2}, \sigma(f^2x, fx), \sigma(f^2x, y), \sigma(f^2x, fy) \right\} \right).$$

Definition 1.3 ([10]). *Let (X, σ) be a metric space and $\mathcal{T} : X \rightarrow X$ be a function. \mathcal{T} is known as F-weak contraction on (X, σ) if there exist $F \in \mathfrak{S}$ and $\gamma > 0$ such that for all $x, y \in X$,*

$$\sigma(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \gamma + F(\sigma(\mathcal{T}x, \mathcal{T}y)) \leq F \left(\max \left\{ \sigma(x, y), \sigma(\mathcal{T}x, x), \sigma(y, \mathcal{T}y), \frac{\sigma(x, \mathcal{T}y) + \sigma(y, \mathcal{T}x)}{2} \right\} \right).$$

Definition 1.4 ([10]). *Let (X, σ) be a metric space and $\mathcal{T} : X \rightarrow X$ be a function. \mathcal{T} is known as δ -weak contraction on (X, σ) if there exist $\delta : [0, \infty) \rightarrow [0, \infty)$ such that for all $x, y \in X$, $\sigma(\mathcal{T}x, \mathcal{T}y) \leq \sigma(x, y) - \delta(\sigma(x, y))$.*

Definition 1.5 ([1]). *The function $\mu : [0, \infty) \times [0, \infty) \rightarrow \mathcal{R}$ is said to be a simulation function, if the following properties hold:*

$$(\mu_1) \quad \mu(0, 0) = 0;$$

$$(\mu_2) \quad \mu(a, b) < a - b \text{ for all } a, b > 0;$$

(μ_3) If $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \{a_n\} = \lim_{n \rightarrow \infty} \{b_n\} = \ell, \text{ then } \limsup_{n \rightarrow \infty} \mu(a_n, b_n) < 0.$$

Let \prod be the class of all simulation functions $\mu : [0, \infty) \times [0, \infty) \rightarrow \mathcal{R}$. In this paper, we established fixed point results by making several combinations of $\sigma(x, y)$ in the context of complete metric space by virtue of simulation function. As application, various analogous results in fixed point theory are easily deduced by conferring different values to s .

2. Main results.

Definition 2.1. Let (X, σ) be a metric space, $S : X \rightarrow X$ be a mapping and $\mu \in \Pi$. Then, S is known as modified weak contraction in cubic terms with respect to μ if the following condition is fulfilled.

$$(2.1) \quad \mu[(1 + s\sigma(x, y))\sigma^2(Sx, Sy), sM(x, y) + L(x, y) - \delta(L(x, y))] \geq 0,$$

where

$$(2.2) \quad L(x, y) = \max \left\{ \sigma^2(x, y), \sigma(x, Sx)\sigma(y, Sy), \sigma(x, Sy)\sigma(y, Sx), \right. \\ \left. \frac{\sigma(x, Sx)\sigma(x, Sy) + \sigma(y, Sx)\sigma(y, Sy)}{2} \right\}$$

and

$$(2.3) \quad M(x, y) = \max \left\{ \frac{\sigma(x, Sx)\sigma(y, Sy) + \sigma(x, Sx)\sigma^2(y, Sy)}{2}, \right. \\ \left. \sigma(x, Sx)\sigma(x, Sy)\sigma(y, Sx), \sigma(x, Sy)\sigma(y, Sx)\sigma(y, Sy) \right\}.$$

where $s \geq 0$ be a real number and $\delta : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\delta(t) = 0$ iff $t = 0$ and $\delta(t) > 0$ for every $t > 0$.

Before starting our Theorem, we will give some lemmas that have an important role for proving our main result.

Lemma 2.2. Let (X, σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ such that the sequence $\{x_n\}$ in X is

defined as $x_{n+1} = Sx_n$ for all $n \in \mathcal{N}$. Let $x_n \neq x_{n+1}$ for each $n \in \mathcal{N}$. Then,

$$\lim_{n \rightarrow \infty} \sigma(x_n, x_{n+1}) = 0.$$

Proof. Firstly, we have to show that $\{\rho_n\}$ is non increasing sequence and convergent. Now, two cases arise.

Case 1. If n is even, substituting $x = x_{2n}$, $y = x_{2n+1}$ in (2.1), we get

$$(2.4) \quad \mu[(1 + s\sigma(x_{2n}, x_{2n+1}))\sigma^2(Sx_{2n}, Sx_{2n+1}), \\ sM(x_{2n}, x_{2n+1}) + L(x_{2n}, x_{2n+1}) - \delta(L(x_{2n}, x_{2n+1}))] \geq 0,$$

where

$$(2.5) \quad L(x_{2n}, x_{2n+1}) = \max \left\{ \sigma^2(x_{2n}, x_{2n+1}), \sigma(x_{2n}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1}), \right. \\ \left. \sigma(x_{2n}, Sx_{2n+1})\sigma(x_{2n+1}, Sx_{2n}), \right. \\ \left. \frac{\sigma(x_{2n}, Sx_{2n})\sigma(x_{2n}, Sx_{2n+1}) + \sigma(x_{2n+1}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1})}{2} \right\}$$

and

$$(2.6) \quad M(x_{2n}, x_{2n+1}) \\ = \max \left\{ \frac{\sigma^2(x_{2n}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1}) + \sigma(x_{2n}, Sx_{2n})\sigma^2(x_{2n+1}, Sx_{2n+1})}{2}, \right. \\ \left. \sigma(x_{2n}, Sx_{2n})\sigma(x_{2n}, Sx_{2n+1})\sigma(x_{2n+1}, Sx_{2n}), \right. \\ \left. \sigma(x_{2n}, Sx_{2n+1})\sigma(x_{2n+1}, Sx_{2n})\sigma(x_{2n+1}, Sx_{2n+1}) \right\}.$$

Using the given assumption of lemma, (2.4), (2.5) and (2.6) reduces to

$$(2.7) \quad \mu[(1 + s\sigma(x_{2n}, x_{2n+1}))\sigma^2(x_{2n+1}, x_{2n+2}), \\ sM(x_{2n}, x_{2n+1}) + L(x_{2n}, x_{2n+1}) - \delta(L(x_{2n}, x_{2n+1}))] \geq 0,$$

where

$$(2.8) \quad L(x_{2n}, x_{2n+1}) = \max \left\{ \begin{aligned} &\sigma^2(x_{2n}, x_{2n+1}), \sigma(x_{2n}, x_{2n+1})\sigma(x_{2n+1}, x_{2n+2}), \\ &\sigma(x_{2n}, x_{2n+2})\sigma(x_{2n+1}, x_{2n+1}), \\ &\frac{\sigma(x_{2n}, x_{2n+1})\sigma(x_{2n}, x_{2n+2}) + \sigma(x_{2n+1}, x_{2n+1})\sigma(x_{2n+1}, x_{2n+2})}{2} \end{aligned} \right\}$$

and

$$(2.9) \quad \begin{aligned} M(x_{2n}, x_{2n+1}) \\ = \max \left\{ \begin{aligned} &\frac{\sigma(x_{2n}, x_{2n+1})\sigma(x_{2n+1}, x_{2n+2}) + \sigma(x_{2n}, x_{2n+1})\sigma^2(x_{2n+1}, x_{2n+2})}{2}, \\ &\sigma(x_{2n}, x_{2n+1})\sigma(x_{2n}, x_{2n+2})\sigma(x_{2n+1}, x_{2n+1}), \\ &\sigma(x_{2n}, x_{2n+2})\sigma(x_{2n+1}, x_{2n+1})\sigma(x_{2n+1}, x_{2n+2}) \end{aligned} \right\}. \end{aligned}$$

Let $\rho_{2n} = \sigma(x_{2n}, x_{2n+1})$. Now, (2.7), (2.8) and (2.9) reduces to

$$(2.10) \quad \mu[(1+s\rho_{2n})\rho_{2n+1}, sM(x_{2n}, x_{2n+1}) + L(x_{2n}, x_{2n+1}) - \delta(L(x_{2n}, x_{2n+1}))] \geq 0,$$

where

$$(2.11) \quad L(x_{2n}, x_{2n+1}) = \max \left\{ \rho_{2n}^2, \rho_{2n+1}\rho_{2n}, 0, \frac{\rho_{2n}(\sigma(x_{2n}, x_{2n+2}))}{2}, 0 \right\}$$

and

$$(2.12) \quad M(x_{2n}, x_{2n+1}) = \max \left\{ \frac{\rho_{2n}^2\rho_{2n+1} + \rho_{2n}\rho_{2n+1}^2}{2}, 0, 0 \right\}.$$

On account of triangle inequality, we infer that

$$\sigma(x_{2n}, x_{2n+1}) + \sigma(x_{2n+1}, x_{2n+2}) \geq \sigma(x_{2n}, x_{2n+2}).$$

Therefore,

$$\sigma(x_{2n}, x_{2n+2}) \leq \rho_{2n+1} + \rho_{2n}.$$

Taking(2.11) into account, we obtain

$$(2.13) \quad L(x_{2n}, x_{2n+1}) \leq \max \left\{ \rho_{2n}^2, \rho_{2n+1}\rho_{2n}, 0, \frac{\rho_{2n}(\rho_{2n} + \rho_{2n+1})}{2}, 0 \right\}.$$

If $\rho_{2n+1} \geq \rho_{2n}$ then (2.10) yields that

$$\begin{aligned} 0 &\leq \mu[s\rho_{2n+1}^2, s\rho_{2n+1}^2 - \delta(\rho_{2n+1}^2)] \\ &< s\rho_{2n+1}^2 - \delta(\rho_{2n+1}^2) - s\rho_{2n+1}^2. \end{aligned}$$

Therefore,

$$s\rho_{2n+1}^2 + \delta(\rho_{2n+1}^2) < s\rho_{2n+1}^2,$$

a contradiction.

Hence, $\rho_{2n+1} \leq \rho_{2n}$. Similarly, we can prove that $\rho_{2n+2} \leq \rho_{2n+1}$ when n is odd natural number. Consequently, ρ_n is decreasing sequence.

Therefore, there exist $q \geq 0$ such that

$$\lim_{n \rightarrow \infty} \rho_n = q.$$

Assume that $q > 0$, then substituting $x = x_n$ and $y = x_{n+1}$ in the inequality (2.1), (2.2) and (2.3), we get

$$(2.14) \quad \mu[(1 + s\sigma(x_n, x_{n+1}))\sigma^2(Sx_n, Sx_{n+1}), \\ sM(x_n, x_{n+1}) + L(x_n, x_{n+1}) - \delta(L(x_n, x_{n+1}))] \geq 0,$$

where

$$(2.15) \quad L(x_n, x_{n+1}) = \max \left\{ \sigma^2(x_n, x_{n+1}), \sigma(x_n, Sx_n)\sigma(x_{n+1}, Sx_{n+1}), \right. \\ \left. \sigma(x_n, Sx_{n+1})\sigma(x_{n+1}, Sx_n), \right. \\ \left. \frac{\sigma(x_n, Sx_n)\sigma(x_n, Sx_{n+1}) + \sigma(x_{n+1}, Sx_n)\sigma(x_{n+1}, Sx_{n+1})}{2} \right\}$$

and

$$(2.16) \quad M(x_n, x_{n+1}) \\ = \max \left\{ \frac{\sigma(x_n, Sx_n)\sigma(x_{n+1}, Sx_{n+1}) + \sigma(x_n, Sx_n)\sigma^2(x_{n+1}, Sx_{n+1})}{2}, \right. \\ \left. \sigma(x_n, Sx_n)\sigma(x_n, Sx_{n+1})\sigma(x_{n+1}, Sx_n), \right. \\ \left. \sigma(x_n, Sx_{n+1})\sigma(x_{n+1}, Sx_n)\sigma(x_{n+1}, Sx_{n+1}) \right\}.$$

By using the given assumption of lemma, (2.14), (2.15) and (2.16) yields that

$$(2.17) \quad \mu[(1 + s\sigma(x_n, x_{n+1}))\sigma^2(x_{n+1}, x_{n+2}), \\ sM(x_n, x_{n+1}) + L(x_n, x_{n+1}) - \delta(L(x_n, x_{n+1}))] \geq 0,$$

where

$$L(x_n, x_{n+1}) = \max \left\{ \sigma^2(x_n, x_{n+1}), \sigma(x_n, x_{n+1})\sigma(x_{n+1}, x_{n+2}), \right. \\ \left. \sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1}), \right. \\ \left. \frac{\sigma(x_n, x_{n+1})\sigma(x_n, x_{n+2}) + \sigma(x_{n+1}, x_{n+1})\sigma(x_{n+1}, x_{n+2})}{2} \right\}$$

and

$$M(x_n, x_{n+1}) = \max \left\{ \frac{\sigma(x_n, x_{n+1})\sigma(x_{n+1}, x_{n+2}) + \sigma(x_n, x_{n+1})\sigma^2(x_{n+1}, x_{n+2})}{2}, \right. \\ \left. \sigma(x_n, x_{n+1})\sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1}), \sigma(x_n, x_{n+2})\sigma(x_{n+1}, x_{n+1})\sigma(x_{n+1}, x_{n+2}) \right\}.$$

Making $n \rightarrow \infty$ in (2.17), we obtain

$$0 \leq \mu[(1 + sq)q^2, sq^3 + q^2 - \delta(q^2)] \\ < sq^3 + q^2 - \delta(q^2) - (1 + sq)q^2.$$

Therefore,

$$(1 + sq)q^2 < (1 + sq)q^2 - \delta(q^2).$$

Due to positivity of q , $\delta(q^2) \leq 0$.

Further, using the property of δ , we get $q = 0$. Consequently,

$$\lim_{n \rightarrow \infty} \sigma(x_n, x_{n+1}) = 0. \quad \square$$

Lemma 2.3. *Let (X, σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ . Let $\{x_n\}$ be the sequence in X such that $x_{n+1} = Sx_n$ with $x_n \neq x_{n+1}$ for all $n \in \mathcal{N}$. Then the picard sequence $\{x_n\}$ is bounded.*

Proof. Let $x_0 \in X$ be any point. Let us suppose that $\{x_n\}$ is not a bounded sequence. Therefore, there exists a subsequence $\{x_{m_\ell}\}$ such that $m_1 = 1$ and $m_{\ell+1}$ is smallest integer for every positive integer ℓ such that

$$\sigma(x_{m_{\ell+1}}, x_{m_\ell}) > 1$$

and

$$\sigma(x_n, x_{m_\ell}) \leq 1$$

for $m_\ell \leq n \leq m_{\ell+1} - 1$.

On account of triangle inequality, we get

$$\begin{aligned} 1 &\leq \sigma(x_{m_{\ell+1}}, x_{m_\ell}) \\ &\leq \sigma(x_{m_{\ell+1}}, x_{m_{\ell+1}-1}) + \sigma(x_{m_{\ell+1}-1}, x_{m_\ell}) \\ &\leq \sigma(x_{m_{\ell+1}}, x_{m_{\ell+1}-1}) + 1. \end{aligned}$$

Making $\ell \rightarrow \infty$ and using lemma 2.2, we obtain

$$\lim_{\ell \rightarrow \infty} \sigma(x_{m_{\ell+1}}, x_{m_\ell}) = 1.$$

Since, S is modified weak contraction in cubic terms with respect to $\mu \in \Pi$, therefore by using μ_3 , we have

$$(2.18) \quad 0 \leq \limsup_{\ell \rightarrow \infty} \mu[(1 + s\sigma(x_{m_{\ell+1}-1}, x_{m_\ell-1}))\sigma^2(Sx_{m_{\ell+1}-1}, Sx_{m_\ell-1}), \\ sM(x_{m_{\ell+1}-1}, x_{m_\ell-1}) + L(x_{m_{\ell+1}-1}, x_{m_\ell-1}) - \delta(L(x_{m_{\ell+1}-1}, x_{m_\ell-1}))] < 0,$$

where

$$\begin{aligned} &L(x_{m_{\ell+1}-1}, x_{m_\ell-1}) \\ &= \max \left\{ \sigma^2(x_{m_{\ell+1}-1}, x_{m_\ell-1}), \sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_\ell-1}, Sx_{m_\ell}), \right. \\ &\quad \left. \sigma(x_{m_{\ell+1}-1}, Sx_{m_\ell-1})\sigma(x_{m_\ell-1}, Sx_{m_{\ell+1}-1}), \right. \\ &\quad \left. \frac{\sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_{\ell+1}-1}, Sx_{m_\ell-1}) + \sigma(x_{m_\ell-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_\ell-1}, Sx_{m_\ell-1})}{2} \right\} \end{aligned}$$

and

$$\begin{aligned}
 & M(x_{m_{\ell+1}-1}, x_{m_{\ell}-1}) \\
 &= \max \left\{ \sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell}-1})\sigma(x_{m_{\ell}-1}, Sx_{m_{\ell+1}-1}), \right. \\
 &\quad \left. \sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell}-1})\sigma(x_{m_{\ell}-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_{\ell}-1}, Sx_{m_{\ell}-1}), \right. \\
 &\quad \left. \frac{\sigma^2(x_{m_{\ell+1}-1}, Sx_{m_{\ell+1}-1})\sigma(x_{m_{\ell}-1}, Sx_{m_{\ell}-1}) + \sigma(x_{m_{\ell+1}-1}, Sx_{m_{\ell+1}-1})\sigma^2(x_{m_{\ell}-1}, Sx_{m_{\ell}-1})}{2} \right\}.
 \end{aligned}$$

Contradiction(2.18) proves our result. \square

Lemma 2.4. *Let (X, σ) be a metric space and S be modified weak contraction in cubic terms with respect to μ . Let $\{x_n\}$ be the sequence in X such that $x_{n+1} = Sx_n$ with $x_n \neq x_{n+1}$ for all $n \in \mathcal{N}$. Then the picard sequence $\{x_n\}$ is a Cauchy sequence.*

Proof. Let us suppose that $\{x_n\}$ is not a Cauchy sequence. Let $\delta_1 > 0$ be however small for which we have sequences $\{n_\ell\}$ and $\{m_\ell\}$ such that

$$\sigma(x_{n_\ell}, x_{m_\ell-1}) < \delta_1$$

and

$$\sigma(x_{n_\ell}, x_{m_\ell}) \geq \delta_1,$$

for each $m_\ell > n_\ell > \ell$.

On account of triangle inequality, we get

$$\begin{aligned}
 (2.19) \quad \delta_1 &\leq \sigma(x_{n_\ell}, x_{m_\ell}) \\
 &\leq \sigma(x_{n_\ell}, x_{m_\ell-1}) + \sigma(x_{m_\ell-1}, x_{m_\ell}).
 \end{aligned}$$

Making $\ell \rightarrow \infty$ in (2.19), we get

$$(2.20) \quad \lim_{\ell \rightarrow \infty} \sigma(x_{n_\ell}, x_{m_\ell}) = \delta_1.$$

Again using triangle inequality, we obtain

$$\sigma(x_{n_\ell}, x_{n_{\ell+1}}) \geq | \sigma(x_{m_\ell}, x_{n_{\ell+1}}) - \sigma(x_{n_\ell}, x_{m_\ell}) |.$$

Letting $\ell \rightarrow \infty$ in the above inequality and taking lemma 2.2, (2.20) into account, we get

$$(2.21) \quad \lim_{\ell \rightarrow \infty} \sigma(x_{m_\ell}, x_{n_{\ell+1}}) = \delta_1.$$

With the help of triangle inequality, we obtain

$$\sigma(x_{m_\ell}, x_{m_\ell+1}) \geq | \sigma(x_{n_\ell}, x_{m_\ell+1}) - \sigma(x_{n_\ell}, x_{m_\ell}) |.$$

Letting $\ell \rightarrow \infty$ in the above inequality and using lemma 2.2, (2.20), we get

$$(2.22) \quad \lim_{\ell \rightarrow \infty} \sigma(x_{n_\ell}, x_{m_\ell+1}) = \delta_1.$$

Now, using triangle inequality, we get

$$| \sigma(x_{n_\ell}, x_{n_\ell+1}) + \sigma(x_{m_\ell}, x_{m_\ell+1}) | \geq | \sigma(x_{n_\ell+1}, x_{m_\ell+1}) - \sigma(x_{n_\ell}, x_{m_\ell}) |.$$

Letting $\ell \rightarrow \infty$ in the above inequality and using lemma 2.2, (2.20), we get

$$(2.23) \quad \lim_{\ell \rightarrow \infty} \sigma(x_{m_\ell+1}, x_{n_\ell+1}) = \delta_1.$$

Substituting $x = x_{n_\ell}$ and $y = x_{m_\ell}$ in (2.1), we get

$$(2.24) \quad \mu[(1 + s\sigma(x_{n_\ell}, x_{m_\ell}))\sigma^2(Sx_{n_\ell}, Sx_{m_\ell}), \\ sM(x_{n_\ell}, x_{m_\ell}) + L(x_{n_\ell}, x_{m_\ell}) - \delta(L(x_{n_\ell}, x_{m_\ell}))] \geq 0,$$

where

$$(2.25) \quad L(x_{n_\ell}, x_{m_\ell}) = \max \left\{ \sigma^2(x_{n_\ell}, x_{m_\ell}), \sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell}), \right. \\ \left. \sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell}), \right. \\ \left. \frac{\sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{n_\ell}, Sx_{m_\ell}) + \sigma(x_{m_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell})}{2} \right\}$$

and

$$(2.26) \quad M(x_{n_\ell}, x_{m_\ell}) \\ = \max \left\{ \frac{\sigma^2(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell}) + \sigma(x_{n_\ell}, Sx_{n_\ell})\sigma^2(x_{m_\ell}, Sx_{m_\ell}), \right. \\ \left. \sigma(x_{n_\ell}, Sx_{n_\ell})\sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell}), \right. \\ \left. \sigma(x_{n_\ell}, Sx_{m_\ell})\sigma(x_{m_\ell}, Sx_{n_\ell})\sigma(x_{m_\ell}, Sx_{m_\ell}) \right\}.$$

Using the given assumption of lemma, (2.24), (2.25) and (2.26) reduces to

$$(2.27) \quad \mu[(1 + s\sigma(x_{n_\ell}, x_{m_\ell}))\sigma^2(x_{n_\ell+1}, x_{m_\ell+1}), \\ sM(x_{n_\ell}, x_{m_\ell}) + L(x_{n_\ell}, x_{m_\ell}) - \delta(L(x_{n_\ell}, x_{m_\ell}))] \geq 0,$$

where

$$L(x_{n_\ell}, x_{m_\ell}) = \max \left\{ \sigma^2(x_{n_\ell}, x_{m_\ell}), \sigma(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1}), \right. \\ \sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1}), \\ \left. \frac{\sigma(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{n_\ell}, x_{m_\ell+1}) + \sigma(x_{m_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1})}{2} \right\}$$

and

$$M(x_{n_\ell}, x_{m_\ell}) \\ = \max \left\{ \frac{\sigma^2(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1}) + \sigma(x_{n_\ell}, x_{n_\ell+1})\sigma^2(x_{m_\ell}, x_{m_\ell+1}), \right. \\ \sigma(x_{n_\ell}, x_{n_\ell+1})\sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1}), \\ \left. \sigma(x_{n_\ell}, x_{m_\ell+1})\sigma(x_{m_\ell}, x_{n_\ell+1})\sigma(x_{m_\ell}, x_{m_\ell+1}) \right\}.$$

Letting $\ell \rightarrow \infty$ in (2.27) and using (2.20), (2.21), (2.22), (2.23), we get

$$0 \leq \mu \left[(1 + s\delta_1)\delta_1^2, s \max \left\{ \frac{0+0}{2}, 0, 0 \right\} + \delta_1^2 - \delta(\delta_1^2) \right] \\ = \mu [(1 + s\delta_1)\delta_1^2, \delta_1^2 - \delta(\delta_1^2)] \\ < \delta_1^2 - \delta(\delta_1^2) - (1 + s\delta_1)\delta_1^2,$$

which implies that

$$(1 + s\delta_1)\delta_1^2 < \delta_1^2 - \delta(\delta_1^2),$$

which is a contradiction.

Consequently, $\{x_n\}$ is a cauchy sequence in X . \square

Theorem 2.5. *Let (X, σ) be a complete metric space and S be modified weak contraction in cubic terms with respect to μ such that the sequence $\{x_n\}$ in X is defined as $x_{n+1} = Sx_n$ for all $n \in \mathcal{N}$. Let $x_n \neq x_{n+1}$ for each $n \in \mathcal{N}$. Then, S has a unique fixed point in X .*

Proof. Since, (X, σ) be a complete metric space and in lemma (2.4), we proved that $\{x_n\}$ is a cauchy sequence in X . Therefore, it yields that

$$(2.28) \quad \lim_{m \rightarrow \infty} x_m = v,$$

for some $v \in X$.

Now, we shall prove that v is fixed point of S . Let us assume contrary that $v \neq Sv$. So, $\sigma(Sv, v) > 0$. Now, substituting $x = x_m$ and $y = v$ in (2.1), we have

$$(2.29) \quad 0 \leq \lim_{m \rightarrow \infty} \sup \mu[(1 + s\sigma(x_m, v))\sigma^2(Sx_m, Sv), \\ sM(x_m, v) + L(x_m, v) - \delta(L(x_m, v))],$$

where

$$(2.30) \quad L(x_m, v) = \max \left\{ \sigma^2(x_m, v), \sigma(x_m, Sx_m)\sigma(v, Sv), \sigma(x_m, Sv)\sigma(v, Sx_m), \right. \\ \left. \frac{\sigma(x_m, Sx_m)\sigma(x_m, Sv) + \sigma(v, Sx_m)\sigma(v, Sv)}{2} \right\}$$

and

$$(2.31) \quad M(x_m, v) = \max \left\{ \frac{\sigma^2(x_m, Sx_m)\sigma(v, Sv) + \sigma(x_m, Sx_m)\sigma^2(v, Sv)}{2}, \right. \\ \left. \sigma(x_m, Sx_m)\sigma(x_m, Sv)\sigma(v, Sx_m), \sigma(x_m, Sv)\sigma(v, Sx_m)\sigma(v, Sv) \right\}.$$

Due to given hypothesis of Theorem and using (2.28), we obtain

$$(2.32) \quad 0 \leq \lim_{m \rightarrow \infty} \sup \mu[(1 + s\sigma(v, v))\sigma^2(v, Sv), sM(v, v) + L(v, v) - \delta(L(v, v))] < 0,$$

where

$$L(v, v) = \max \left\{ \sigma^2(v, v), \sigma(v, v)\sigma(v, Sv), \sigma(v, Sv)\sigma(v, v), \right. \\ \left. \frac{\sigma(v, v)\sigma(v, Sv) + \sigma(v, v)\sigma(v, Sv)}{2} \right\}$$

and

$$M(v, v) = \max \left\{ \frac{\sigma^2(v, v)\sigma(v, Sv) + \sigma(v, v)\sigma^2(v, Sv)}{2}, \right. \\ \left. \sigma(v, v)\sigma(v, Sv)\sigma(v, v), \sigma(v, Sv)\sigma(v, v)\sigma(v, Sv) \right\}.$$

The inequality (2.32) is a contradiction. Therefore, $\sigma(Sv, v) = 0$, which implies that $Sv = v$. Hence, S has a fixed point v in X .

Now, we examine the uniqueness of attained fixed point of S . Let us assume that v and w be two fixed points of S . Putting $x = v$ and $y = w$ in (2.1), we can derive that $\sigma(v, w) = 0$. Therefore, $v = w$.

Consequently, S has unique fixed point in X . \square

Example 2.6. Consider $X = \{0, 1, 2, 3\}$ associated with the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ |x - y| & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Now, we define the mapping $S : X \rightarrow X$ by

$$Sx = \begin{cases} 1 & \text{if } x = 3 \\ 0 & \text{otherwise} \end{cases}.$$

Let $\delta : [0, +\infty) \rightarrow [0, +\infty)$ be defined as $\delta(t) = \frac{t}{3}$. Indeed, for every value of $s > 0$ and $x, y \in X$, it is simple to justify that the condition of inequality(2.1) fulfils. So, we guarantee the occurrence of fixed point of S . Therefore, Theorem (2.5) holds.

Corollary 2.7. *Let (X, σ) be a complete metric space and S be self map which fulfils the condition*

$$\mu[\sigma^2(Sx, Sy), L(x, y) - \delta(L(x, y))] \geq 0,$$

where

$$L(x, y) = \max \left\{ \sigma^2(x, y), \sigma(x, Sx)\sigma(y, Sy), \sigma(x, Sy)\sigma(y, Sx), \right. \\ \left. \frac{\sigma(x, Sx)\sigma(x, Sy) + \sigma(y, Sx)\sigma(y, Sy)}{2} \right\},$$

where $\delta : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\delta(t) = 0$ iff $t = 0$ and $\delta(t) > 0$ for every $t > 0$. Then, S has a unique fixed point in X .

Proof. By substituting $s = 0$ in Theorem 2.5, we will get the unique fixed point of S . \square

Corollary 2.8. *Let (X, σ) be a complete metric space and S be self map which fulfils the condition*

$$\mu[(1 + 2\sigma(x, y))\sigma^2(Sx, Sy), 2M(x, y) + L(x, y) - \delta(L(x, y))] \geq 0,$$

where

$$L(x, y) = \max \left\{ \sigma^2(x, y), \sigma(x, Sx)\sigma(y, Sy), \sigma(x, Sy)\sigma(y, Sx), \right. \\ \left. \frac{\sigma(x, Sx)\sigma(x, Sy) + \sigma(y, Sx)\sigma(y, Sy)}{2} \right\}$$

and

$$M(x, y) = \max \left\{ \frac{\sigma(x, Sx)\sigma(y, Sy) + \sigma(x, Sx)\sigma^2(y, Sy)}{2}, \right. \\ \left. \sigma(x, Sx)\sigma(x, Sy)\sigma(y, Sx), \sigma(x, Sy)\sigma(y, Sx)\sigma(y, Sy) \right\},$$

where $\delta : [0, \infty) \rightarrow [0, \infty)$ is a continuous function with $\delta(t) = 0$ iff $t = 0$ and $\delta(t) > 0$ for every $t > 0$. Then, S has a unique fixed point in X .

Proof. By substituting $s = 2$ in Theorem 2.5, we will get the unique fixed point of S . \square

3. Conclusion. This paper explore the new notion of modified weak contraction in cubic terms of metric via simulation functions and established fixed point results for this class of contraction in the setting of complete metric space. The results obtained herein represent a significant extension of the existing δ -weak contraction framework, broadening its applicability by incorporating cubic terms of the metric and the powerful analytical tool of simulation functions. The fixed point theorems presented in this work not only generalize several prominent results in the existing literature but also open new directions for further investigation in metric fixed point theory. The illustrative example provided confirms the validity and applicability of our theoretical results.

4. Open problem. It is an open problem that whether the foundational framework established in this research can be extended to modular metric space, b -metric space, partial metric space and rectangular metric space.

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