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SOME OPERATIONS ON CONGRUENCE LABELED GRAPHS

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ABSTRACT. A graph $G(V, E)$ is identified as congruence graph, if the vertices and edges are labeled by distinctive natural numbers, which induces $f(x_i) \equiv f(x_j) \pmod{g(y)}$, where each edge $x_i x_j \in E$ with $x_i, x_j \in V$. In this paper, some operations were performed on congruence graphs such as union of congruence path graphs & $C_m \cup P_n$ are identified as congruence graph. Furthermore, graphs procured by duplicating any arbitrary vertex, edge in a cycle C_n and cartesian product of two path are confirmed as congruence graph.

1. Introduction. The assignment of natural numbers to the vertices and edges of a graph with some constraint is called graph labeling. The technique of labeling was developed in the 1960s. Alexander Rose [8] introduced three types

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of labeling, namely α, β and ρ labeling. Later, Solomon Golomb renamed β-labeling as graceful labeling. Graham and Sloane [4] introduced harmonic graph labeling, and Bloom and Hsu [1] included the graceful labeling technique for directed graphs. Gallian [3] reviewed numerous methods, all of which inspired young researchers and influenced them to apply labeling techniques in various fields such as database management, networks, rocket coding, astronomy, and many others.

In the recent years enormous articles have been published in graph labeling, since it is the best method to explore the situation and maintain the optimal solution. Congruence labeling was introduced and recognized several graphs, such as paths, cycles, and friendship graphs, as congruence graphs by the authors in [6]. This paper is devoted to verifying some unary and binary operations, such as union, duplication of vertices or edges, and Cartesian product, on graphs labeled for congruence as congruence graphs. Some results based on operations on congruence graphs are also defined.

2. Preliminaries.

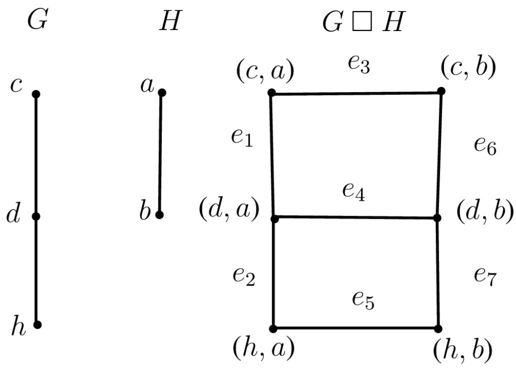
Definition 2.1 ([5]). *The number of edges in G connected to a vertex is defined as the* degree *of the vertex.*

Definition 2.2 ([3]). *A graph G' constructed by duplicating an arbitrary vertex v of G is defined as a duplicate graph, where v' is the duplicate vertex with $N\left(v'\right) = N\left(v\right)$.*

Definition 2.3 ([6]). *Let $G = (V, E)$ be a graph with $|V| = r$ and $|E| = s$ be defined as a* congruence graph *if there exists a vertex labeling $f : V \to \{1, 2, \ldots, K\}$, $x_i \in V$ and induces the edge labeling $g : E \to \{1, 2, \ldots, K-1\}$, provided that $f\left(x_{i+1}\right) \equiv f\left(x_i\right) \pmod{g\left(y_i\right)}$, for each $y_i = x_i x_{i+1}$, where $K = \min\{2r, 2s\}$.*

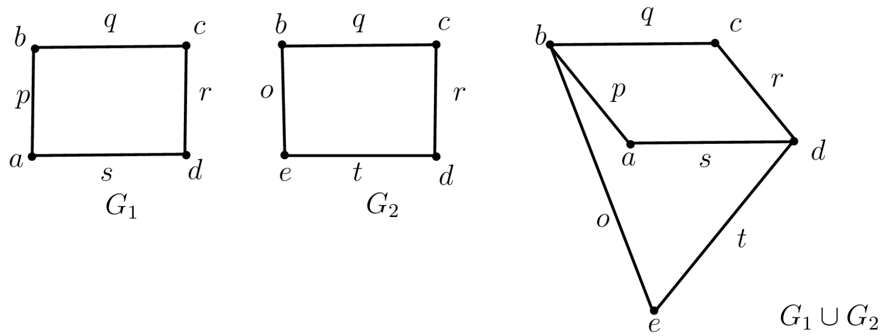
Definition 2.4 ([10]). *The* Cartesian product *of G and H is $G \,\square\, H$, where $V\left(G \times H\right) = V\left(G\right) \times V\left(H\right)$ and $\left(a, c\right)\left(b, d\right)$ is an edge if $a = b$ and $cd \in E\left(H\right)$ or $c = d$ and $ab \in E\left(G\right)$.*

Example 2.5.



Definition 2.6 ([2])**.** $G_1 \cup G_2$, *is a* union of graphs G_1 *and* G_2, *such that* $V(G_1) \cup V(G_2)$ *and* $E(G_1) \cup E(G_2)$ *are the set of vertices and the set of edges, respectively.*

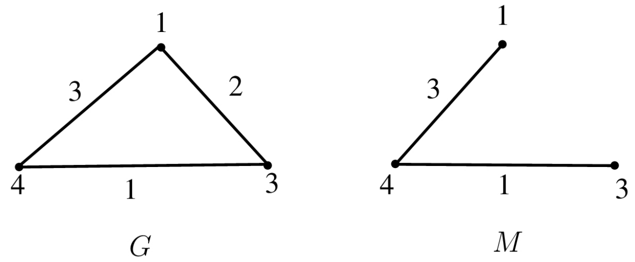
Example 2.7.



3. Main Results.

Definition 3.1. *A subgraph M of a congruence graph G is congruent as a* subcongruence graph *if* $V(M) \subseteq V(G)$ *and* $E(M) \subseteq E(G)$.

Example 3.2.



Theorem 3.3. *Every congruence graph has at least one subcongruence graph.*

Proof. Suppose that $G(V,E)$ is a congruence graph, where $V(G) = \{x_1, x_2, \ldots, x_r\}$ and $E(G) = \{y_1, y_2, \ldots, y_s\}$, respectively.

Let x_1 and x_2 be any two vertices of G whose labels are a_1 and a_2, respectively, and y_1 be the edge connecting these two vertices, labeled as b_1 based on modular mathematical division.

Removing edge y_1 results in a new graph $S = G - y_1$ with $|V(S)| = r$ and $|E(S)| = s - 1$. The labels of subgraph S will remain as in G. Therefore, it is a congruence graph. Thus, every congruence graph has at least one subcongruence graph. □

Theorem 3.4. *$P_r \cup P_r$ is a congruence graph, for $r \geq 1$.*

Proof. Suppose $G = P_r \cup P_r$ with $|V(G)| = 2r$ and $|E(G)| = 2r - 2$. G is disconnected and each of its components is a path. G admits congruence labeling. Since each component of G admits congruence labeling. Therefore, the union of two congruent path graphs is a congruence graph. □

Corollary. The union of a finite number of graphs of congruence paths is a congruence graph. The union of isomorphic congruence graphs is a congruence graph.

Theorem 3.5. *The union of any two congruence graph is a congruence graph.*

Proof. The union of the congruence graphs G_1 and G_2 is G, where $|V(G_1)| = r_1$, $|V(G_2)| = r_2$, $|E(G_1)| = s_1$ and $|E(G_2)| = s_2$.

Case 1: Suppose G is disconnected, then $V(G_1)$, $V(G_2)$ have no common vertex.

$$|V(G)| = r_1 + r_2 \quad \text{and} \quad |E(G)| = s_1 + s_2,$$

here $K = \min(2(r_1 + r_2), 2(s_1 + s_2))$.

The labels of the first component are preserved and second one is labeled in the following way. Suppose $r_1 < r_2$, $2r_1$ will be the last vertex label of G_1, and G_2 is labeled with $2r_1 + 1, 2r_1 + 2, \ldots, 2(r_1 + r_2)$ if $r_2 < s_2$, otherwise to $2(s_1 + s_2)$. If $r_1 > s_1$, $2s_1$ is the last label of G_1 and G_2 is labeled by $2s_1 + 1, 2s_1 + 2, \ldots, 2(s_1 + r_2)$ if $r_2 < s_2$, otherwise by $2(s_1 + s_2)$. Therefore, G is a congruence graph.

Case 2: Suppose G is connected, then G_1 and G_2 have at least one common vertex, i.e. have similar label

$$|V(G)| < r_1 + r_2 \quad \text{and} \quad |E(G)| \leq s_1 + s_2,$$

here $K = \min(2(r_1 + r_2 - 1), 2(s_1 + s_2))$.

The vertices and edges of G from G_1 are preserved with their labels, and the rest are labeled as specified in the previous case. Therefore, the union of the congruence graph is a congruence graph. □

Theorem 3.6. *$P_{r_1} \cup C_{r_2}$ is a congruence graph for $r_1 \geq 2$, $r_2 \geq 3$.*

Proof. Let $G = P_{r_1} \cup C_{r_2}$ with $|V(G)| = r_1 + r_2$ and $|E(G)| = r_1 + r_2 - 1$, where,

$$\begin{aligned} V(P_{r_1}) &= x_1, x_2, \ldots, x_{r_1}, & V(C_{r_2}) &= w_1, w_2, \ldots, w_{r_2}, \\ E(P_{r_1}) &= y_1, y_2, \ldots, y_{r_1 - 1}, & E(C_{r_2}) &= t_1, t_2, \ldots, t_{r_2 - 1}, \end{aligned}$$

$K = \min(2(r_1 + r_2), 2(r_1 + r_2 - 1)) = 2(r_1 + r_2 - 1)$.

The labeling of the vertices and edges is given by the functions $f : V \to \{1, 2, 3, \ldots, K\}$ and $g : E \to \{1, 2, 3, \ldots, K - 1\}$, defined as

$$\begin{aligned} g(y_i) &= i + 1 \\ f(x_{r_1}) &= K - 2r_2 \\ f(x_{r_1 - i}) &= \begin{cases} |f(x_{r_1 - i + 1}) - g(y_{r_1 - i})|, & \text{for } i \text{ is odd} \\ |f(x_{r_1 - i + 1}) + g(y_{r_1 - i})|, & \text{for } i \text{ is even.} \end{cases} \end{aligned}$$

Consider two cases:

If r_2 is an even number, then the vertex labeling function is defined as given below

$$f(w_i) = \begin{cases} 1, & \text{for } i = 1 \\ K, & \text{for } i = r_2 \\ K-1, & \text{for } i = r_2 - 1 \\ 2r_2 - \dfrac{i}{2}, & \text{for } i = 2, 4, \ldots, r_2 - 2 \\ \dfrac{i+1}{2}, & \text{for } i = 3, 5, \ldots, r_2 - 3. \end{cases}$$

If r_2 is an odd number, then the vertex are labeled as follows

$$f(w_i) = \begin{cases} 1, & \text{for } i = 1 \\ K, & \text{for } i = r_2 \\ K-1, & \text{for } i = r_2 - 1 \\ 2r_2 - \dfrac{i}{2}, & \text{for } i = 2, 4, \ldots, r_2 - 3 \\ \dfrac{i+1}{2}, & \text{for } i = 3, 5, \ldots, r_2 - 2. \end{cases}$$

$$g(t_i) = \begin{cases} |f(w_i) - f(w_{i+1})|, & \text{for } i \leq (r_2 - 1) \\ [4pt] |f(w_{i+1}) - f(w_1)|, & \text{for } i = r_2 \end{cases}$$

The labels of each y_i and $t_i \in E$ establish the modulus of correspondence of integers to the labels of the vertices.

Therefore, the union of the cycle and path graph $P_{r_1} \cup C_{r_2}$ is a congruence graph for $r_1 \geq 2$, $r_2 \geq 3$. □

Example 3.7. Let us consider the congruence graphs of the cycle P_5 and the path C_6, examining its union $C_6 \cup P_5$. Here $K = 20$. The congruence labeling of $P_5 \cup C_6$ is shown in Fig. 1.

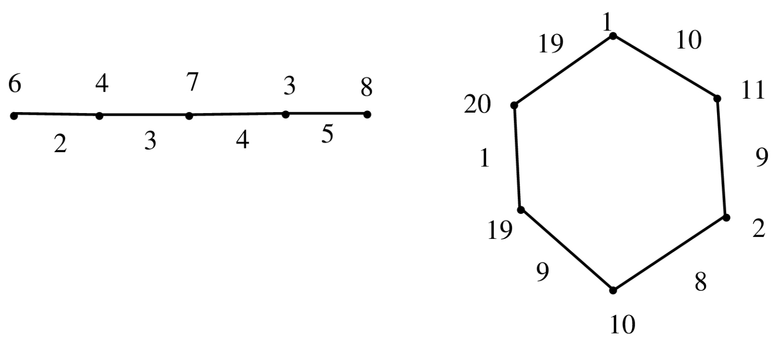


Fig. 1. $P_6 \cup C_5$ congruence graph

Theorem 3.8. *Duplicating an arbitrary vertex of C_r generates a congruence graph for $r \geq 3$.*

Proof. Let $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$ be the vertices and edges of C_r. Suppose G' is the new graph generated after duplicating an arbitrary vertex of C_r.

Let x_1 be duplicated and the newly added vertex be x_1'. The new set of vertices and edges of G are $\left(x_1, x_2, \ldots, x_n, x_1'\right)$ and $\left(y_1, y_2, \ldots, y_n, y_1', y_n'\right)$, respectively.

$$K = \min\left(2\left(r+1\right), 2\left(r+2\right)\right) = 2\left(r+1\right).$$

The vertex and edge labeling functions are outlined as follows.

$$f\left(x_1'\right) = K$$
$$g\left(y_1'\right) = 1$$
$$g\left(y_r'\right) = r-1$$

If r is an even number, then

$$f\left(x_i\right) = \begin{cases} K - r + 1, & \text{for } i = r \\ K - \dfrac{i}{2}, & \text{for } i = 2, 4, \ldots, r-2 \\ \dfrac{i+1}{2}, & \text{for } i = 1, 3, 5, \ldots, r-1 \end{cases}$$

$$g\left(y_i\right) = \begin{cases} K - i - 1, & \text{for } i = 1, 2, \ldots, r-2 \\ K - i - \dfrac{r}{2}, & \text{for } i = r-1 \\ K - r, & \text{for } i = r. \end{cases}$$

If r is an odd number, then

$$f\left(x_i\right) = \begin{cases} K - r + 1, & \text{for } i = r \\ K - \dfrac{i}{2}, & \text{for } i = 2, 4, \ldots, r-1 \\ \dfrac{i+1}{2}, & \text{for } i = 1, 3, 5, \ldots, r-2 \end{cases}$$

$$g(y_i) = \begin{cases} K - i - 1, & \text{for } i = 1, 2, \ldots, r-2 \\ \frac{i}{2}, & \text{for } i = r-1 \\ K - r, & \text{for } i = r. \end{cases}$$

Therefore, duplicating any vertex of C_r generates a congruence graph. □

Example 3.9. A duplication of an arbitrary vertex from C_{10} is shown. Here $K = 22$. The congruence labeling of a vertex duplication in C_{10} is discussed in Fig. 2.

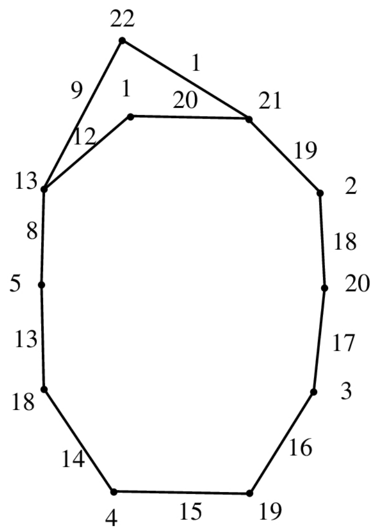


Fig. 2. Duplication of a vertex in C_{10} – congruence graph

Theorem 3.10. *Duplicating an arbitrary edge of C_r generates a congruence graph, $r \geq 4$.*

Proof. Suppose G is the graph generated by duplicating an arbitrary edge of C_r. Let this be y_1 and the newly added vertices and edges are x_1', x_2', y_1', y_2' and y_r'. The sets of vertices and edges of G are $(x_1, x_2, \ldots, x_r, x_1', x_2')$ and $(y_1, y_2, \ldots, y_r, y_1', y_2', y_r')$, respectively.

$$K = \min(2(r+2), 2(r+3)) = 2(r+2).$$

The vertex and edge labeling functions are defined as follows

$$f(x_1') = K - r + 1$$

$$f\left(x_2'\right) = K - r - 1$$
$$g\left(y_1'\right) = 2$$
$$g\left(y_2'\right) = r + 1$$
$$g\left(y_r'\right) = 1.$$

If r is an even number

$$f\left(x_i\right) = \begin{cases} K - r, & \text{for } i = r \\ K - \frac{i}{2}, & \text{for } i = 2, 4, \ldots, r-2 \\ \frac{i+1}{2}, & \text{for } i = 1, 3, 5, \ldots, r-1 \end{cases}$$

$$g\left(y_i\right) = \begin{cases} K - i - 1, & \text{for } i = 1, 2, \ldots, r-2 \\ K - i - 1 - \frac{n}{2}, & \text{for } i = r-1 \\ K - r - 1, & \text{for } i = r. \end{cases}$$

If r is an odd number, then

$$f\left(x_i\right) = \begin{cases} K - r, & \text{for } i = r \\ K - \frac{i}{2}, & \text{for } i = 2, 4, \ldots, r-1 \\ \frac{i+1}{2}, & \text{for } i = 1, 3, 5, \ldots, r-2 \end{cases}$$

$$g\left(y_i\right) = \begin{cases} K - i - 1, & \text{for } i = 1, 2, \ldots, r-2 \\ 1 + \frac{i}{2}, & \text{for } i = r-1 \\ K - r - 1, & \text{for } i = r. \end{cases}$$

Therefore, duplicating an arbitrary edge of C_r generates a congruence graph. □

Example 3.11. The edge duplication of C_8 is verified. Here $K = 18$. The congruence labeling of the edge duplication in C_8 is shown in Fig. 3.

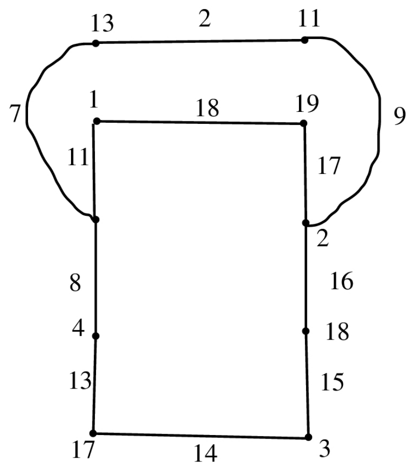


Fig. 3. Duplication of an edge in C_8 – congruence graph

Theorem 3.12. *The Cartesian product of the two-path graph $P_{r_1} \square P_{r_2}$, is a congruence graph.*

Proof. Suppose $G = P_{r_1} \square P_{r_2}$ with $|V| = r_1 r_2$ and $|E| = 2r_1r_2 - r_1 - r_2$. The vertices are represented as $x_i w_j$ for $1 \le i \le r_1$ and $1 \le j \le r_2$.

$$K = \min\left(2\left(r_1 r_2\right), 2\left(2r_1r_2 - r_1 - r_2\right)\right) = 2\left(r_1 r_2\right).$$

The edges are labeled with $g(y_i) = K - (i+1)$, for every i. The vertices are labeled with

$$f\left(x_i w_j\right) = \begin{cases} K - \left(\dfrac{i+1}{2}\right), & \text{for } i \text{ is odd } \& \ j = 1 \\ \left(\dfrac{i}{2}\right), & \text{for } i \text{ is even } \& \ j = 1 \\ \left(r_1 - 1\right) + \left(\dfrac{i+1}{2}\right), & \text{for } i \text{ is odd } \& \ j = 2 \\ \left(K - r_1 + 1\right) - \left(\dfrac{i+2}{2}\right), & \text{for } i \text{ is even } \& \ j = 2 \\ f\left(x_i w_{j-2}\right) - \left(2r_1 - 1\right), & \text{for } 2f\left(x_i w_{j-2}\right) > K, \\ & \quad i = 1, 2, \ldots, r_1 \ \& \ j = 3, 4, \ldots, r_2 \\ f\left(x_i w_{j-2}\right) + \left(2r_1 - 1\right), & \text{for } 2f\left(x_i w_{j-2}\right) < K, \\ & \quad i = 1, 2, \ldots, r_1 \ \& \ j = 3, 4, \ldots, r_2. \end{cases}$$

Therefore, the Cartesian product of the two-path graph $P_{r_1} \square P_{r_2}$ is a congruence graph $m, n \ge 4$. $\square$

Example 3.13. Consider the Cartesian product of the two-path graph $P_6 \square P_5$. Here $n = 6$, $m = 5$, $K = 60$. The congruence labeling of the Cartesian product of the two-path graph $P_6 \square P_5$ is shown in Fig. 4.

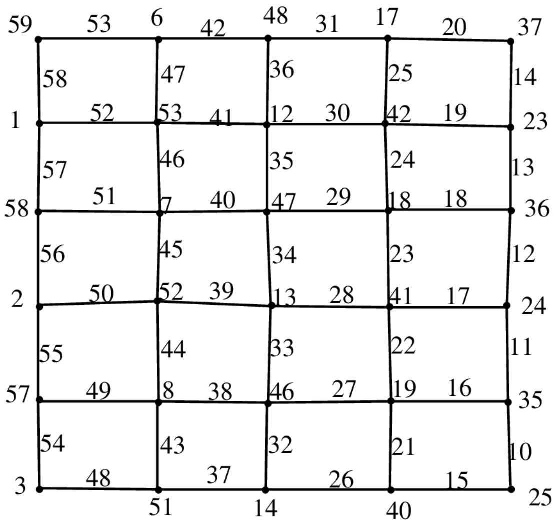


Fig. 4. The Cartesian product of the two-path graph $P_6 \square P_5$ – congruence graph

4. Conclusion. Although all real-world situations are represented as graphs, they will only make sense if they are properly labeled. Labeling graphs helps to distinguish each vertex and edge. This paper discusses some operations on graphs labeled as congruent, such as the existence of a subcongruence graph, the union of graphs of congruence paths & $C_m \cup P_n$, and identification as a congruence graph. In addition, a graph obtained by duplicating an arbitrary vertex, an edge in C_r, and the Cartesian product of two paths are verified as a congruence graph.

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